Neutrosophic logics on Non-Archimedean Structures

Andrew Schumann

May 30, 2009

Abstract

We present a general way that allows to construct systematically analytic calculi for a large family of non-Archimedean many-valued logics: hyperrational-valued, hyperreal-valued, and p-adic valued logics characterized by a special format of semantics with an appropriate rejection of Archimedes' axiom. These logics are built as different extensions of standard many-valued logics (namely, Łukasiewicz's, Gödel's, Product, and Post's logics). The informal sense of Archimedes' axiom is that anything can be measured by a ruler. Also logical multiple-validity without Archimedes' axiom consists in that the set of truth values is infinite and it is not well-founded and well-ordered. We consider two cases of non-Archimedean multi-valued logics: the first with many-validity in the interval $[0, 1]$ of hypernumbers and the second with many-validity in the ring $\mathbf{Z}_p$ of p-adic integers. On the base of non-Archimedean valued logics, we construct non-Archimedean valued interval neutrosophic logics by which we can describe neutrality phenomena.

Belarusian State University, Minsk, Belarus
e-mail: Andrew.Schumann@gmail.com

1 Introduction

The development of fuzzy logic and fuzziness was motivated in large measure by the need for a conceptual framework which can address the issue of uncertainty and lexical imprecision. Recall that fuzzy logic was introduced by L. Zadeh in 1965 (see [58]) to represent data and information possessing nonstatistical uncertainties. Florentin Smarandache had generalized fuzzy logic and introduced two new concepts (see [51], [52], [53]):

1. neutrosophy as study of neutralities;
2. neutrosophic logic and neutrosophic probability as a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, etc.

Neutrosophy proposed by Smarandache in [53] is a new branch of philosophy, which studies the nature of neutralities, as well as their logical applications. The notion of neutrality is explicated within the framework of neutrosophic logic introduced by Smarandache in [51], [52].

In this paper we consider neutrosophic logic as a generalization of non-Archimedean valued logic, where the truth values of ${}^*[0,1]$ are extended to truth triples of the form $\langle t, i, f\rangle \subseteq ({}^*[0,1])^3$, where t is the truth-degree, i the indeterminacy-degree, f the falsity-degree and they are approximated by non-standard subsets of ${}^*[0,1]$, and these subsets may overlap and exceed the unit interval in the sense of the non-Archimedean analysis.

Neutrosophic logic is an alternative to all the existing logics, because it represents a mathematical model of uncertainty on non-Archimedean structures. It is a non-classical logic in which each proposition is estimated to have the percentage of truth in a subset $t \subseteq {}^*[0,1]$, the percentage of indeterminacy in a subset $i \subseteq {}^*[0,1]$, and the percentage of falsity in a subset $f \subseteq {}^*[0,1]$. Thus, neutrosophic logic is a formal frame trying to measure the truth, indeterminacy, and falsehood simultaneously, therefore it generalizes:

- Boolean logic ($i = \emptyset$, t and f consist of either 0 or 1);
- n-valued logic ($i = \emptyset$, t and f consist of members 0, 1, ..., $n-1$);
- fuzzy logic ($i = \emptyset$, t and f consist of members of $[0,1]$).

In simple neutrosophic logic, where t, i, f are singletons, the tautologies have the truth value $\langle {}^*1, {}^*0, {}^*0\rangle$, the contradictions the value $\langle {}^*0, {}^*1, {}^*1\rangle$. While for a paradox, we have the truth value $\langle {}^*1, {}^*1, {}^*1\rangle$. Indeed, the paradox is the only proposition true and false in the same time in the same world, and indeterminate as well! We can assume that some statements are indeterminate in all possible worlds, i.e. that there exists "absolute indeterminacy" $\langle {}^*1, {}^*1, {}^*1\rangle$.

Dezert suggested to develop practical applications of neutrosophic logic (see [54], [55]), e.g. for solving certain practical problems posed in the domain of research in Data/Information fusion.

In the next sections, we shall consider the following non-Archimedean structures:

1. the nonstandard extension ${}^*\mathbf{Q}$ (called the field of hyperrational numbers),
2. the nonstandard extension ${}^*\mathbf{R}$ (called the field of hyperreal numbers),
3. the nonstandard extension $\mathbf{Z}_p$ (called the ring of p-adic integers) that we obtain as follows. Let the set $\mathbf{N}$ of natural numbers be the index set and let $\Theta = \{0, \ldots, p-1\}$. Then the nonstandard extension $\Theta^{\mathbf{N}} \backslash \mathcal{U} = \mathbf{Z}_p$.

Further, we shall set the following logics on non-Archimedean structures:

- hyperrational valued Łukasiewicz's, Gödel's, and Product logics,
- hyperreal valued Łukasiewicz's, Gödel's, and Product logics,
- p-adic valued Łukasiewicz's, Gödel's logics.

Recall that non-Archimedean logical multiple-validities were considered by Schumann in [45] – [49].

2 Hyper-valued logics

Assume that ${}^*\mathbf{Q}_{[0,1]} = \mathbf{Q}_{[0,1]}^{\mathbf{N}}/\mathcal{U}$ is a nonstandard extension of the subset $\mathbf{Q}_{[0,1]} = \mathbf{Q} \cap [0,1]$ of rational numbers, where $\mathcal{U}$ is the FRECHET filter that may be no ultrafilter, and ${}^\sigma\mathbf{Q}_{[0,1]} \subset {}^*\mathbf{Q}_{[0,1]}$ is the subset of standard members. We can extend the usual order structure on $\mathbf{Q}_{[0,1]}$ to a partial order structure on ${}^*\mathbf{Q}_{[0,1]}$:

1. for rational numbers x, $y \in \mathbf{Q}_{[0,1]}$ we have $x \leq y$ in $\mathbf{Q}_{[0,1]}$ iff $[f] \leq [g]$ in ${}^*\mathbf{Q}_{[0,1]}$, where $\{\alpha \in \mathbf{N}\colon f(\alpha) = x\} \in \mathcal{U}$ and $\{\alpha \in \mathbf{N}\colon g(\alpha) = y\} \in \mathcal{U}$,

   i.e., f and g are constant functions such that $[f] = {}^*x$ and $[g] = {}^*y$,

2. each positive rational number ${}^*x \in {}^\sigma\mathbf{Q}_{[0,1]}$ is greater than any number $[f] \in {}^*\mathbf{Q}_{[0,1]} \backslash {}^\sigma\mathbf{Q}_{[0,1]}$,

   i.e., ${}^*x > [f]$ for any positive $x \in \mathbf{Q}_{[0,1]}$ and $[f] \in {}^*\mathbf{Q}_{[0,1]}$, where $[f]$ isn't constant function.

These conditions have the following informal sense:

1. The sets ${}^\sigma\mathbf{Q}_{[0,1]}$ and $\mathbf{Q}_{[0,1]}$ have isomorphic order structure.
2. The set ${}^*\mathbf{Q}_{[0,1]}$ contains actual infinities that are less than any positive rational number of ${}^\sigma\mathbf{Q}_{[0,1]}$.

Define this partial order structure on ${}^*\mathbf{Q}_{[0,1]}$ as follows:

$\mathcal{O}_{{}^*\mathbf{Q}}$

1. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] \leq [g]$ if

$$\{\alpha \in \mathbf{N}\colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}.$$

2. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] < [g]$ if

$$\{\alpha \in \mathbf{N}\colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$$

   and $[f] \neq [g]$, i.e., $\{\alpha \in \mathbf{N}\colon f(\alpha) \neq g(\alpha)\} \in \mathcal{U}$.

3. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] = [g]$ if $f \in [g]$.

This ordering relation is not linear, but partial, because there exist elements $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, which are incompatible.

Introduce two operations max, min in the partial order structure $\mathcal{O}_{{}^*\mathbf{Q}}$:

1. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, $\min([f], [g]) = [f]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{{}^*\mathbf{Q}}$,
2. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, $\max([f], [g]) = [g]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{{}^*\mathbf{Q}}$,
3. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, $\min([f], [g]) = \max([f], [g]) = [f] = [g]$ if and only if $[f] = [g]$ under condition $\mathcal{O}_{{}^*\mathbf{Q}}$,

4. for all hyperrational numbers $[f],[g] \in {}^*\mathbf{Q}_{[0,1]}$, if $[f],[g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{Q}}$, then $\min([f],[g]) = [h]$ iff
$$\{\alpha \in \mathbf{N} \colon \min(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$
5. for all hyperrational numbers $[f],[g] \in {}^*\mathbf{Q}_{[0,1]}$, if $[f],[g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{Q}}$, then $\max([f],[g]) = [h]$ iff
$$\{\alpha \in \mathbf{N} \colon \max(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

It is easily seen that conditions $1-3$ are corollaries of conditions 4, 5.

Note there exist the maximal number ${}^*1 \in {}^*\mathbf{Q}_{[0,1]}$ and the minimal number ${}^*0 \in {}^*\mathbf{Q}_{[0,1]}$ under condition $\mathcal{O}_{^*\mathbf{Q}}$. Therefore, for any $[f] \in {}^*\mathbf{Q}_{[0,1]}$, we have: $\max({}^*1,[f]) = {}^*1$, $\max({}^*0,[f]) = [f]$, $\min({}^*1,[f]) = [f]$ and $\min({}^*0,[f]) = {}^*0$.

Now define *hyperrational-valued* ŁUKASIEWICZ*'s logic* $\mathfrak{M}_{^*\mathbf{Q}}$:

Definition 1 *The ordered system* $\langle V_{^*\mathbf{Q}}, \neg_L, \rightarrow_L, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{{}^*1\}\rangle$ *is called hyperrational valued* ŁUKASIEWICZ*'s matrix logic* $\mathfrak{M}_{^*\mathbf{Q}}$*, where*

1. $V_{^*\mathbf{Q}} = {}^*\mathbf{Q}_{[0,1]}$ *is the subset of hyperrational numbers,*
2. *for all* $[x] \in V_{^*\mathbf{Q}}$, $\neg_L[x] = {}^*1 - [x]$,
3. *for all* $[x],[y] \in V_{^*\mathbf{Q}}$, $[x] \rightarrow_L [y] = \min({}^*1, {}^*1 - [x] + [y])$,
4. *for all* $[x],[y] \in V_{^*\mathbf{Q}}$, $[x] \vee [y] = ([x] \rightarrow_L [y]) \rightarrow_L [y] = \max([x],[y])$,
5. *for all* $[x],[y] \in V_{^*\mathbf{Q}}$, $[x] \wedge [y] = \neg_L(\neg_L[x] \vee \neg_L[y]) = \min([x],[y])$,
6. *for a subset* $M \subseteq V_{^*\mathbf{Q}}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,
7. *for a subset* $M \subseteq V_{^*\mathbf{Q}}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,
8. $\{{}^*1\}$ *is the set of designated truth values.*

The truth value ${}^*0 \in V_{^*\mathbf{Q}}$ is false, the truth value ${}^*1 \in V_{^*\mathbf{Q}}$ is true, and other truth values $x \in V_{^*\mathbf{Q}} \backslash \{{}^*0, {}^*1\}$ are neutral.

If we replace the set $\mathbf{Q}_{[0,1]}$ by $\mathbf{R}_{[0,1]}$ and the set ${}^*\mathbf{Q}_{[0,1]}$ by ${}^*\mathbf{R}_{[0,1]}$ in all above definitions, then we obtain *hyperreal valued matrix logic* $\mathfrak{M}_{^*\mathbf{R}}$.

Definition 2 *Hyper-valued* GÖDEL*'s matrix logic* $G_{^*[0,1]}$ *is the structure* $\langle {}^*[0,1], \neg_G, \rightarrow_G, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{{}^*1\}\rangle$*, where*

1. *for all* $[x] \in {}^*[0,1]$, $\neg_G[x] = [x] \rightarrow_G {}^*0$,
2. *for all* $[x],[y] \in {}^*[0,1]$, $[x] \rightarrow_G [y] = {}^*1$ *if* $[x] \leqslant [y]$ *and* $[x] \rightarrow_G [y] = [y]$ *otherwise,*
3. *for all* $[x],[y] \in {}^*[0,1]$, $[x] \vee [y] = \max([x],[y])$,
4. *for all* $[x],[y] \in {}^*[0,1]$, $[x] \wedge [y] = \min([x],[y])$,

5. *for a subset* $M \subseteq {}^*[0,1]$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,

6. *for a subset* $M \subseteq {}^*[0,1]$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,

7. $\{{}^*1\}$ *is the set of designated truth values.*

The truth value ${}^*0 \in {}^*[0,1]$ is false, the truth value ${}^*1 \in {}^*[0,1]$ is true, and other truth values $[x] \in {}^*(0,1)$ are neutral.

Definition 3 *Hyper-valued Product matrix logic* $\Pi_{^*[0,1]}$ *is the structure* $\langle {}^*[0,1], \neg_\Pi, \rightarrow_\Pi, \&_\Pi, \wedge, \vee, \widetilde{\exists}, \widetilde{\forall}, \{{}^*1\}\rangle$, *where*

1. *for all* $[x] \in {}^*[0,1]$, $\neg_\Pi[x] = [x] \rightarrow_\Pi {}^*0$,

2. *for all* $[x],[y] \in {}^*[0,1]$, $[x] \rightarrow_\Pi [y] = \begin{cases} {}^*1, & \textit{if } [x] \leqslant [y], \\ \min({}^*1, \frac{[y]}{[x]}), & \textit{otherwise;} \end{cases}$

3. *for all* $[x],[y] \in {}^*[0,1]$, $[x]\&_\Pi[y] = [x] \cdot [y]$,

4. *for all* $[x],[y] \in {}^*[0,1]$, $[x] \wedge [y] = [x] \cdot ([x] \rightarrow_\Pi [y])$,

5. *for all* $[x],[y] \in {}^*[0,1]$, $[x] \vee [y] = (([x] \rightarrow_\Pi [y]) \rightarrow_\Pi [y]) \wedge (([y] \rightarrow_\Pi [x]) \rightarrow_\Pi [x])$,

6. *for a subset* $M \subseteq {}^*[0,1]$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,

7. *for a subset* $M \subseteq {}^*[0,1]$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,

8. $\{{}^*1\}$ *is the set of designated truth values.*

The truth value ${}^*0 \in {}^*[0,1]$ is false, the truth value ${}^*1 \in {}^*[0,1]$ is true, and other truth values $[x] \in {}^*(0,1)$ are neutral.

3 p-Adic valued logics

Let us remember that the expansion

$$n = \alpha_{-N} \cdot p^{-N} + \alpha_{-N+1} \cdot p^{-N+1} + \ldots + \alpha_{-1} \cdot p^{-1} + \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum_{k=-N}^{+\infty} \alpha_k \cdot p^k,$$

where $\alpha_k \in \{0,1,\ldots,p-1\}$, $\forall k \in \mathbf{Z}$, and $\alpha_{-N} \neq 0$, is called the *canonical expansion of p-adic number n* (or p-adic expansion for n). The number n is called p-adic. This number can be identified with sequences of digits: $n = \ldots \alpha_2 \alpha_1 \alpha_0, \alpha_{-1} \alpha_{-2} \ldots \alpha_{-N}$. We denote the set of such numbers by $\mathbf{Q}_p$.

The expansion $n = \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum\limits_{k=0}^{\infty} \alpha_k \cdot p^k$, where $\alpha_k \in \{0,1,\ldots,p-1\}$, $\forall k \in \mathbf{N} \cup \{0\}$, is called the *expansion of p-adic integer*

n. The integer n is called p-adic. This number sometimes has the following notation: $n = \ldots \alpha_3\alpha_2\alpha_1\alpha_0$. We denote the set of such numbers by $\mathbf{Z}_p$.

If $n \in \mathbf{Z}_p$, $n \neq 0$, and its canonical expansion contains only a finite number of nonzero digits α_j, then n is natural number (and vice versa). But if $n \in \mathbf{Z}_p$ and its expansion contains an infinite number of nonzero digits α_j, then n is an infinitely large natural number. Thus the set of p-adic integers contains actual infinities $n \in \mathbf{Z}_p \backslash \mathbf{N}$, $n \neq 0$. This is one of the most important features of non-Archimedean number systems, therefore it is natural to compare $\mathbf{Z}_p$ with the set of nonstandard numbers ${}^*\mathbf{Z}$.

Extend the standard order structure on $\mathbf{N}$ to a partial order structure on $\mathbf{Z}_p$.

- for any $x, y \in \mathbf{N}$ we have $x \leq y$ in $\mathbf{N}$ iff $x \leq y$ in $\mathbf{Z}_p$,
- each finite natural number x is less than any infinite number y, i.e. $x < y$ for any $x \in \mathbf{N}$ and $y \in \mathbf{Z}_p \backslash \mathbf{N}$, $y \neq 0$.

Define this partial order structure on $\mathbf{Z}_p$ as follows:

$\mathcal{O}_{\mathbf{Z}_p}$ Let $x = \ldots x_n \ldots x_1 x_0$ and $y = \ldots y_n \ldots y_1 y_0$ be the canonical expansions of two p-adic integers $x, y \in \mathbf{Z}_p$. (1) We set $x < y$ if the following three conditions hold: (i) there exists n such that $x_n < y_n$; (ii) $x_k \leq y_k$ for all $k > n$; (iii) x is a finite integer, i.e. there exists l such that $x_m = 0$ for all $m \geq l$. (2) We set $x = y$ if $x_n = y_n$ for each $n = 0, 1, \ldots$ (3) Suppose that both x and y are infinite integers. We set $x \leq y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$ and we set $x < y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$ and there exists n_0 such that $x_{n_0} < y_{n_0}$.

Now introduce two operations max, min in the partial order structure on $\mathbf{Z}_p$:

1 for all p-adic integers $x, y \in \mathbf{Z}_p$, $\min(x, y) = x$ if and only if $x \leq y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$,

2 for all p-adic integers $x, y \in \mathbf{Z}_p$, $\max(x, y) = y$ if and only if $x \leq y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$,

3 for all p-adic integers $x, y \in \mathbf{Z}_p$, $\max(x, y) = \min(x, y) = x = y$ if and only if $x = y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$.

The ordering relation $\mathcal{O}_{\mathbf{Z}_p}$ is not linear, but partial, because there exist elements $x, z \in \mathbf{Z}_p$, which are incompatible. As an example, let $p = 2$ and let $x = -\frac{1}{3} = \ldots 10101 \ldots 101$, $z = -\frac{2}{3} = \ldots 01010 \ldots 010$. Then the numbers x and z are incompatible.

Thus,

4 Let $x = \ldots x_n \ldots x_1 x_0$ and $y = \ldots y_n \ldots y_1 y_0$ be the canonical expansions of two p-adic integers $x, y \in \mathbf{Z}_p$ and x, y are incompatible under condition $\mathcal{O}_{\mathbf{Z}_p}$. We get $\min(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set

1. $z_n = y_n$ if $x_n \geq y_n$,
2. $z_n = x_n$ if $x_n \leq y_n$,

3. $z_n = x_n = y_n$ if $x_n = y_n$.

We get $\max(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set

1. $z_n = y_n$ if $x_n \leq y_n$,
2. $z_n = x_n$ if $x_n \geq y_n$,
3. $z_n = x_n = y_n$ if $x_n = y_n$.

It is important to remark that there exists the maximal number $N_{max} \in \mathbf{Z}_p$ under condition $\mathcal{O}_{\mathbf{Z}_p}$. It is easy to see:

$$N_{max} = -1 = (p-1) + (p-1) \cdot p + \ldots + (p-1) \cdot p^k + \ldots = \sum_{k=0}^{\infty} (p-1) \cdot p^k$$

Therefore

5 $\min(x, N_{max}) = x$ and $\max(x, N_{max}) = N_{max}$ for any $x \in \mathbf{Z}_p$.

Now consider *p-adic valued* ŁUKASIEWICZ*'s matrix logic* $\mathfrak{M}_{\mathbf{Z}_p}$.

Definition 4 *The ordered system* $\langle V_{\mathbf{Z}_p}, \neg_L, \rightarrow_L, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{N_{max}\}\rangle$ *is called p-adic valued* ŁUKASIEWICZ*'s matrix logic* $\mathfrak{M}_{\mathbf{Z}_p}$*, where*

1. $V_{\mathbf{Z}_p} = \{0, \ldots, N_{max}\} = \mathbf{Z}_p$,
2. *for all* $x \in V_{\mathbf{Z}_p}$, $\neg_L x = N_{max} - x$,
3. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \rightarrow_L y = (N_{max} - \max(x, y) + y)$,
4. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \vee y = (x \rightarrow_L y) \rightarrow_L y = \max(x, y)$,
5. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \wedge y = \neg_L(\neg_L x \vee \neg_L y) = \min(x, y)$,
6. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\exists}(M) = \max(M)$*, where* $\max(M)$ *is a maximal element of* M*,*
7. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\forall}(M) = \min(M)$*, where* $\min(M)$ *is a minimal element of* M*,*
8. $\{N_{max}\}$ *is the set of designated truth values.*

The truth value $0 \in \mathbf{Z}_p$ is false, the truth value $N_{max} \in \mathbf{Z}_p$ is true, and other truth values $x \in \mathbf{Z}_p \backslash \{0, N_{max}\}$ are neutral.

Definition 5 *p-Adic valued* GÖDEL*'s matrix logic* $G_{\mathbf{Z}_p}$ *is the structure* $\langle V_{\mathbf{Z}_p}$, $\neg_G$, $\rightarrow_G$, $\vee$, $\wedge$, $\widetilde{\exists}$, $\widetilde{\forall}$, $\{N_{max}\}\rangle$*, where*

1. $V_{\mathbf{Z}_p} = \{0, \ldots, N_{max}\} = \mathbf{Z}_p$,
2. *for all* $x \in V_{\mathbf{Z}_p}$, $\neg_G x = x \rightarrow_G 0$,
3. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \rightarrow_G y = N_{max}$ *if* $x \leqslant y$ *and* $x \rightarrow_G y = y$ *otherwise,*
4. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \vee y = \max(x, y)$,

5. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \wedge y = \min(x, y)$,

6. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,

7. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,

8. $\{N_{max}\}$ *is the set of designated truth values.*

4 Non-Archimedean valued BL-algebras

Now introduce the following new operations defined for all $[x], [y] \in {}^*\mathbf{Q}$ in the partial order structure $\mathcal{O}_{^*\mathbf{Q}}$:

- $[x] \rightarrow_L [y] = {}^*1 - \max([x], [y]) + [y]$,
- $[x] \rightarrow_\Pi [y] = {}^*1$ if $[x] \leq [y]$ and $[x] \rightarrow_\Pi [y] = \min({}^*1, \frac{[y]}{[x]})$ otherwise,

  notice that we have $\min({}^*1, \frac{[y]}{[x]}) = [h]$ iff there exists $[h] \in {}^*\mathbf{Q}_{[0,1]}$ such that $\{\alpha \in \mathbf{N}\colon \min(1, \frac{y(\alpha)}{x(\alpha)}) = h(\alpha)\} \in \mathcal{U}$, let us also remember that the members $[x]$, $[y]$ can be incompatible under $\mathcal{O}_{^*\mathbf{Q}}$,
- $\neg_L[x] = {}^*1 - [x]$, i.e. $[x] \rightarrow_L {}^*0$,
- $\neg_\Pi[x] = {}^*1$ if $[x] = {}^*0$ and $\neg_\Pi[x] = {}^*0$ otherwise, i.e. $\neg_\Pi[x] = [x] \rightarrow_\Pi {}^*0$,
- $\Delta[x] = {}^*1$ if $[x] = {}^*1$ and $\Delta[x] = {}^*0$ otherwise, i.e. $\Delta[x] = \neg_\Pi\neg_L[x]$,
- $[x]\&_L[y] = \max([x], {}^*1 - [y]) + [y] - {}^*1$, i.e. $[x]\&_L[y] = \neg_L([x] \rightarrow_L \neg_L[y])$,
- $[x]\&_\Pi[y] = [x] \cdot [y]$,
- $[x] \oplus [y] := \neg_L[x] \rightarrow_L [y]$,
- $[x] \ominus [y] := [x]\&_L\neg_L[y]$,
- $[x] \wedge [y] = \min([x], [y])$, i.e. $[x] \wedge [y] = [x]\&_L([x] \rightarrow_L [y])$,
- $[x] \vee [y] = \max([x], [y])$, i.e. $[x] \vee [y] = ([x] \rightarrow_L [y]) \rightarrow_L [y]$,
- $[x] \rightarrow_G [y] = {}^*1$ if $[x] \leq [y]$ and $[x] \rightarrow_G [y] = [y]$ otherwise, i.e. $[x] \rightarrow_G [y] = \Delta([x] \rightarrow_L [y]) \vee [y]$,
- $\neg_G[x] := [x] \rightarrow_G {}^*0$.

A *hyperrational valued BL-matrix* is a structure $\mathbf{L}_{^*\mathbf{Q}} = \langle {}^*\mathbf{Q}_{[0,1]}, \wedge, \vee, *, \Rightarrow, {}^*0, {}^*1\rangle$ such that (1) $\langle {}^*\mathbf{Q}_{[0,1]}, \wedge, \vee, {}^*0, {}^*1\rangle$ is a lattice with the largest element ^{*1} and the least element ^{*0}, (2) $\langle {}^*\mathbf{Q}_{[0,1]}, *, {}^*1\rangle$ is a commutative semigroup with the unit element ^{*1}, i.e. $*$ is commutative, associative, and ${}^*1 * [x] = [x]$ for all $[x] \in {}^*\mathbf{Q}_{[0,1]}$, (3) the following conditions hold

$$[z] \leq ([x] \Rightarrow [y]) \text{ iff } [x] * [z] \leq [y] \text{ for all } [x], [y], [z];$$

$$[x] \wedge [y] = [x] * ([x] \Rightarrow [y]);$$

$$[x] \vee [y] = (([x] \Rightarrow [y]) \Rightarrow [y]) \wedge (([y] \Rightarrow [x]) \Rightarrow [x]),$$

$$([x] \Rightarrow [y]) \vee ([y] \Rightarrow [x]) = {}^*1.$$

If we replace the set $\mathbf{Q}_{[0,1]}$ by $\mathbf{R}_{[0,1]}$ and the set ${}^*\mathbf{Q}_{[0,1]}$ by ${}^*\mathbf{R}_{[0,1]}$ in all above definitions, then we obtain *hyperreal valued BL-matrix* $\mathbf{L}_{^*\mathbf{R}}$. Matrices $\mathbf{L}_{^*\mathbf{Q}}$, $\mathbf{L}_{^*\mathbf{R}}$ are different versions of a non-Archimedean valued BL-algebra. Continuing in the same way, we can build non-Archimedean valued Ł-algebra, $\mathbf{G}$-algebra, and $\mathbf{\Pi}$-algebra.

Further consider the following new operations defined for all $x, y \in \mathbf{Z}_p$ in the partial order structure $\mathcal{O}_{\mathbf{Z}_p}$:

- $x \to_L y = N_{max} - \max(x, y) + y$,
- $x \to_\Pi y = N_{max}$ if $x \leq y$ and $x \to_\Pi y =$ integral part of $\frac{y}{x}$ otherwise,
- $\neg_L x = N_{max} - x$, i.e. $x \to_L 0$,
- $\neg_\Pi x = N_{max}$ if $x = 0$ and $\neg_\Pi x = 0$ otherwise, i.e. $\neg_\Pi x = x \to_\Pi 0$,
- $\Delta x = N_{max}$ if $x = N_{max}$ and $\Delta x = 0$ otherwise, i.e. $\Delta x = \neg_\Pi \neg_L x$,
- $x \&_L y = \max(x, N_{max} - y) + y - N_{max}$, i.e. $x \&_L y = \neg_L (x \to_L \neg_L y)$,
- $x \&_\Pi y = x \cdot y$,
- $x \oplus y := \neg_L x \to_L y$,
- $x \ominus y := x \&_L \neg_L y$,
- $x \wedge y = \min(x, y)$, i.e. $x \wedge y = x \&_L (x \to_L y)$,
- $x \vee y = \max(x, y)$, i.e. $x \vee y = (x \to_L y) \to_L y$,
- $x \to_G y = N_{max}$ if $x \leq y$ and $x \to_G y = y$ otherwise, i.e. $x \to_G y = \Delta(x \to_L y) \vee y$,
- $\neg_G x := x \to_G 0$.

A *p-adic valued BL-matrix* is a structure $\mathbf{L}_{\mathbf{Z}_p} = \langle \mathbf{Z}_p, \wedge, \vee, *, \Rightarrow, 0, N_{max} \rangle$.

5 Non-Archimedean valued predicate logical language

Recall that for each $i \in [0, 1]$, ${}^*i = [f = i]$, i.e. it is a constant function. Every element of ${}^*[0, 1]$ has the form of infinite tuple $[f] = \langle y_0, y_1, \ldots \rangle$, where $y_i \in [0, 1]$ for each $i = 0, 1, 2, \ldots$

Let $\mathcal{L}$ be a standard first-order language associated with p-valued (resp. infinite-valued) semantics. Then we can get an extension $\mathcal{L}'$ of first-order language $\mathcal{L}$ to set later a language of p-adic valued (resp. hyper-valued) logic.

In $\mathcal{L}'$ we build infinite sequences of well-formed formulas of $\mathcal{L}$:

$$\psi^\infty = \langle \psi_1, \ldots, \psi_N, \ldots \rangle,$$

$$\psi^i = \langle \psi_1, \ldots, \psi_i \rangle,$$

where $\psi_j \in \mathcal{L}$.

A formula ψ^∞ (resp. ψ^i) is called a *formula of infinite length* (resp. a *formula of i-th length*).

Definition 6 *Logical connectives in hyper-valued logic are defined as follows:*

1. $\psi^\infty \star \varphi^\infty = \langle \psi_1 \star \varphi_1, \ldots, \psi_N \star \varphi_N, \ldots \rangle$, *where* $\star \in \{\&, \rightarrow\}$;
2. $\neg \psi^\infty = \langle \neg \psi_1, \ldots, \neg \psi_N, \ldots \rangle$;
3. $\mathrm{Q}x\, \psi^\infty = \langle \mathrm{Q}x\, \psi_1, \ldots, \mathrm{Q}x\, \psi_N, \ldots \rangle$, $\mathrm{Q} \in \{\forall, \exists\}$;
4. $\psi^\infty \star \varphi^1 = \langle \psi_1 \star \varphi, \psi_2 \star \varphi, \ldots, \psi_N \star \varphi, \ldots \rangle$, *where* $\star \in \{\&, \rightarrow\}$.

Definition 7 *Logical connectives in p-adic valued logic are defined as follows:*

1. $\psi^\infty \star \varphi^\infty = \langle \psi_1 \star \varphi_1, \ldots, \psi_N \star \varphi_N, \ldots \rangle$, *where* $\star \in \{\&, \rightarrow\}$;
2. $\neg \psi^\infty = \langle \neg \psi_1, \ldots, \neg \psi_N, \ldots \rangle$;
3. $\mathrm{Q}x\, \psi^\infty = \langle \mathrm{Q}x\, \psi_1, \ldots, \mathrm{Q}x\, \psi_N, \ldots \rangle$, $\mathrm{Q} \in \{\forall, \exists\}$.
4. $\psi^\infty \star \varphi^i = \langle \psi_1 \star \varphi_1,\ \ldots,\ \psi_i \star \varphi_i,\ \psi_{i+1} \star \bot,\ \psi_{i+2} \star \bot,\ \ldots,\ \psi_N \star \bot, \ldots \rangle$, *where* $\star \in \{\&, \rightarrow\}$.
5. *suppose* $i < j$, *then* $\psi^i \star \varphi^j = \langle \psi_1 \star \varphi_1,\ \ldots,\ \psi_i \star \varphi_i,\ \psi_{i+1} \star \bot,\ \psi_{i+2} \star \bot,\ \ldots,\ \psi_j \star \bot \rangle$, *where* $\star \in \{\&, \rightarrow\}$.

An interpretation for a language $\mathcal{L}'$ is defined in the standard way. Extend the valuation of $\mathcal{L}$ to one of $\mathcal{L}'$ as follows.

Definition 8 *Given an interpretation* $I = \langle \mathcal{M}, s \rangle$ *and a valuation* val_I *of* $\mathcal{L}$, *we define the non-Archimedean i-valuation* val_I^i *(resp.* ∞*-valuation* val_I^∞*) to be a mapping from formulas* φ^i *(resp.* φ^∞*) of* $\mathcal{L}'$ *to truth value set* V^i *(resp.* ${}^*V$*) as follows:*

1. $\mathrm{val}_I^i(\varphi^i) = \langle \mathrm{val}_I(\varphi_1), \ldots, \mathrm{val}_I(\varphi_i) \rangle$.
2. $\mathrm{val}_I^\infty(\varphi^\infty) = \langle \mathrm{val}_I(\varphi_1), \ldots, \mathrm{val}_I(\varphi_N), \ldots \rangle$.

For example, in p-adic valued case $\mathrm{val}_I^\infty(\psi^\infty \star \varphi^i) = \langle \mathrm{val}_I(\psi_1 \star \varphi_1),\ \ldots,\ \mathrm{val}_I(\psi_i \star \varphi_i),\ \mathrm{val}_I(\psi_{i+1} \star \bot),\ \mathrm{val}_I(\psi_{i+2} \star \bot),\ \ldots,\ \mathrm{val}_I(\psi_N \star \bot), \ldots \rangle$, where $\star \in \{\&, \rightarrow\}$.

Let $\mathbf{L}_{^*V}$ be a non-Archimedean valued BL-matrix. Then the valuations val_I^i and val_I^∞ of $\mathcal{L}'$ to non-Archimedean valued BL-matrix gives the basic fuzzy logic with the non-Archimedean valued semantics.

We say that an $\mathbf{L}_{^*V}$-structure $\mathcal{M}$ is an *i-model* (resp. an ∞-*model*) of an $\mathcal{L}'$-theory T iff $\mathrm{val}_I^i(\varphi^i) = \langle 1, \ldots, 1 \rangle$ (resp. $\mathrm{val}_I^\infty(\varphi^\infty) = {}^*1$) on $\mathcal{M}$ for each $\varphi^i \in T$ (resp. $\varphi^\infty \in T$).

6 Non-Archimedean valued basic fuzzy propositional logic BL_∞

Let us construct a non-Archimedean extension of basic fuzzy propositional logic BL denoted by BL_∞. This logic is built in the language $\mathcal{L}'$ and it has a non-Archimedean valued BL-matrix as its semantics.

Remember that the logic BL has just two propositional operations: &, $\rightarrow$, which are understood as t-norm and its residuum respectively.

The logic BL_∞ is given by the following axioms:

$$(\varphi^i \rightarrow \psi^i) \rightarrow ((\psi^i \rightarrow \chi^i) \rightarrow (\varphi^i \rightarrow \chi^i)), \tag{1}$$

$$(\varphi^i \& \psi^i) \rightarrow \varphi^i, \tag{2}$$

$$(\varphi^i \& \psi^i) \rightarrow (\psi^i \& \varphi^i), \tag{3}$$

$$(\varphi^i \& (\varphi^i \rightarrow \psi^i)) \rightarrow (\psi^i \& (\psi^i \rightarrow \varphi^i)), \tag{4}$$

$$(\varphi^i \rightarrow (\psi^i \rightarrow \chi^i)) \rightarrow ((\varphi^i \& \psi^i) \rightarrow \chi^i), \tag{5}$$

$$((\varphi^i \& \psi^i) \rightarrow \chi^i) \rightarrow (\varphi^i \rightarrow (\psi^i \rightarrow \chi^i)), \tag{6}$$

$$((\varphi^i \rightarrow \psi^i) \rightarrow \chi^i) \rightarrow (((\psi^i \rightarrow \varphi^i) \rightarrow \chi^i) \rightarrow \chi^i), \tag{7}$$

$$\bot^i \rightarrow \psi^i, \tag{8}$$

$$(\varphi^\infty \rightarrow \psi^\infty) \rightarrow ((\psi^\infty \rightarrow \chi^\infty) \rightarrow (\varphi^\infty \rightarrow \chi^\infty)), \tag{9}$$

$$(\varphi^\infty \& \psi^\infty) \rightarrow \varphi^\infty, \tag{10}$$

$$(\varphi^\infty \& \psi^\infty) \rightarrow (\psi^\infty \& \varphi^\infty), \tag{11}$$

$$(\varphi^\infty \& (\varphi^\infty \rightarrow \psi^\infty)) \rightarrow (\psi^\infty \& (\psi^\infty \rightarrow \varphi^\infty)), \tag{12}$$

$$(\varphi^\infty \rightarrow (\psi^\infty \rightarrow \chi^\infty)) \rightarrow ((\varphi^\infty \& \psi^\infty) \rightarrow \chi^\infty), \tag{13}$$

$$((\varphi^\infty \& \psi^\infty) \rightarrow \chi^\infty) \rightarrow (\varphi^\infty \rightarrow (\psi^\infty \rightarrow \chi^\infty)), \tag{14}$$

$$((\varphi^\infty \rightarrow \psi^\infty) \rightarrow \chi^\infty) \rightarrow (((\psi^\infty \rightarrow \varphi^\infty) \rightarrow \chi^\infty) \rightarrow \chi^\infty), \tag{15}$$

$$\bot^\infty \rightarrow \psi^\infty. \tag{16}$$

These axioms are said to be *horizontal*. Introduce also some new axioms that show basic properties of non-Archimedean ordered structures. These express a connection between formulas of various length.

1. **Non-Archimedean multiple-validity**. It is well known that there exist infinitesimals that are less than any positive number of $[0, 1]$. This property can be expressed by means of the following logical axiom:

$$(\neg(\psi^1 \leftrightarrow \psi^\infty)\&\neg(\varphi^1 \leftrightarrow \bot^\infty)) \to (\psi^\infty \to \varphi^1), \tag{17}$$

   where $\psi^1 = \psi_1$, i.e. it is the first member of an infinite tuple ψ^∞.

2. p**-adic multiple-validity**. There is a well known theorem according to that every equivalence class a for which $|a|_p \leq 1$ (this means that a is a p-adic integer) has exactly one representative CAUCHY sequence $\{a_i\}_{i\in\omega}$ for which:

   (a) $0 \leq a_i < p^i$ for $i = 1, 2, 3, \ldots$;
   (b) $a_i \equiv a_{i+1} \mod p^i$ for $i = 1, 2, 3, \ldots$

   This property can be expressed by means of the following logical axioms:

$$\begin{gathered}((\overline{p^{i+1}-1} \ominus \overline{p^i-1}) \to_L \psi^{i+1}) \to_L \\ (\psi^{i+1} \leftrightarrow (\underbrace{\overline{p-1} \oplus \cdots \oplus \overline{p-1}}_{p^i} \oplus \psi^i),\end{gathered} \tag{18}$$

$$\begin{gathered}(\psi^{i+1} \leftrightarrow (\overbrace{\overline{k} \oplus \cdots \oplus \overline{k}}^{p^i}) \oplus \psi^i) \to_L \\ ((((\ldots(\overline{p^{i+1}-1} \ominus \underbrace{\overline{p^i-1}) \ominus \ldots) \ominus \overline{p^i-1})}_{0<p-k\leq p} \\ \ominus\neg_L\overline{k}) \to_L \psi^{i+1}),\end{gathered} \tag{19}$$

$$\begin{gathered}(\psi^{i+1} \leftrightarrow (\overbrace{\overline{k} \oplus \cdots \oplus \overline{k}}^{p^i}) \oplus \psi^i) \to_L \\ (\psi^{i+1} \to_L (((\ldots(\overline{p^{i+1}-1} \ominus \\ \underbrace{\overline{p^i-1}) \ominus \ldots) \ominus \overline{p^i-1})}_{0\leq(p-1)-k\leq p-1} \ominus\neg_L\overline{k})),\end{gathered} \tag{20}$$

$$(\psi^{i+1} \to_L \overline{p^i-1}) \to_L (\psi^{i+1} \leftrightarrow \psi^i), \tag{21}$$

$$\begin{gathered}(\psi^{i+1} \leftrightarrow \psi^i) \vee (\psi^{i+1} \leftrightarrow (\psi^i \oplus p^i \cdot \overline{1})) \vee \ldots \\ \vee(\psi^{i+1} \leftrightarrow (\psi^i \oplus p^i \cdot \overline{p-1})),\end{gathered} \tag{22}$$

   where $\overline{p-1}$ is a tautology at the first-order level and $\overline{p^i-1}$ (respectively $\overline{p^{i+1}-1}$) a tautology for formulas of i-th length (respectively of $(i+1)$-th length); $\psi^1 = \psi_1$, i.e. it is the first member of an infinite tuple ψ^∞; $\neg_L\overline{k}$ is a first-order formula that has the truth value $((p-1)-k) \in \{0,\ldots,p-1\}$ for any its interpretations and $\overline{k}$ is a first-order formula that has the truth value $k \in \{0,\ldots,p-1\}$ for any its interpretations; $\overline{1}$ is a first-order formula that has the truth value 1 for any its interpretations, etc. The denoting $p^i \cdot \overline{k}$ means $\underbrace{\overline{k} \oplus \cdots \oplus \overline{k}}_{p^i}$.

Axioms (17) – (22) are said to be *vertical*.

The deduction rules of BL_∞ is modus ponens: from ψ, $\psi \to \varphi$ infer φ.

The notions of proof, derivability $\vdash$, theorem, and theory over BL_∞ is defined as usual.

Theorem 1 (Soundness and Completeness) *Let Φ be a formula of $\mathcal{L}'$, T an $\mathcal{L}'$-theory. Then the following conditions are equivalent:*

- $T \vdash \Phi$*;*
- $\mathrm{val}^i_I(\Phi) = \langle \underbrace{1,\ldots,1}_{i} \rangle$ *(resp.* $\mathrm{val}^\infty_I(\Phi) = {}^*1$*) for each* $\mathbf{L}_{^*V}$*-model* $\mathcal{M}$ *of* T*;*

Proof. This follows from theorem 4 and semantic rules of BL_∞. □

7 Neutrosophic sets

Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_n\}$, with a generic element of U denoted by u_i. A vague set A in U is characterized by a truth-membership function t_A and a false-membership function f_A,

$$t_A \colon U \to [0,1],$$
$$f_A \colon U \to [0,1],$$

where $t_A(u_i)$ is a lower bound on the grade of membership of u_i derived from the evidence for u_i, $f_A(u_i)$ is a lower bound on the negation of u_i derived from the evidence against u_i, and $t_A(u_i) + f_A(u_i) \leqslant 1$. The grade of membership of u_i in the vague set A is bounded to a subinterval $[t_A(u_i), 1 - f_A(u_i)]$ of $[0,1]$. The vague value $[t_A(u_i), 1 - f_A(u_i)]$ indicates that the exact grade of membership $\mu_A(u_i)$ of u_i may be unknown. But it is bounded by $t_A(u_i) \leqslant \mu_A(u_i) \leqslant 1 - f_A(u_i)$, where $t_A(u_i) + f_A(u_i) \leqslant 1$. When the universe of discourse U is continuous, a vague set A can be written as

$$A = \int_U [t_A(u_i), 1 - f_A(u_i)]/u_i, \quad u_i \in U.$$

When U is discrete, then

$$A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i, \quad u_i \in U.$$

Logical operations in vague set theory are defined as follows:

Let x and y be two vague values, $x = [t_x, 1 - f_x]$, $y = [t_y, 1 - f_y]$, where $t_x \in [0,1]$, $f_x \in [0,1]$, $t_y \in [0,1]$, $f_y \in [0,1]$, $t_x + f_x \leqslant 1$ and $t_y + f_y \leqslant 1$. Then

$$\neg_L x = [1 - t_x, f_x],$$
$$x \wedge y = [\min(t_x, t_y), \min(1 - f_x, 1 - f_y)],$$
$$x \vee y = [\max(t_x, t_y), \max(1 - f_x, 1 - f_y)].$$

8 Neutrosophic set operations

Definition 9 *Let U be the universe of discourse, $U = \{u_1, u_2, \dots, u_n\}$. A hyper-valued neutrosophic set A in U is characterized by a truth-membership function t_A, an indeterminacy-membership function i_A, and a false-membership function f_A*

$$t_A \ni f \colon U \to {}^*[0,1],$$
$$i_A \ni f \colon U \to {}^*[0,1],$$
$$f_A \ni f \colon U \to {}^*[0,1],$$

where t_A is the degree of truth-membership function, i_A is the degree of indeterminacy-membership function, and f_A is the degree of falsity-membership function. There is no restriction on the sum of t_A, i_A, and f_A, i.e.

$$^*0 \leqslant \max t_A(u_i) + \max i_A(u_i) + \max f_A(u_i) \leqslant {}^*3.$$

Definition 10 *Let U be the universe of discourse, $U = \{u_1, u_2, \dots, u_n\}$. A p-adic valued neutrosophic set A in U is characterized by a truth-membership function t_A, an indeterminacy-membership function i_A, and a false-membership function f_A*

$$t_A \ni f \colon U \to \mathbf{Z}_p,$$
$$i_A \ni f \colon U \to \mathbf{Z}_p,$$
$$f_A \ni f \colon U \to \mathbf{Z}_p,$$

where t_A is the degree of truth-membership function, i_A is the degree of indeterminacy-membership function, and f_A is the degree of falsity-membership function. There is no restriction on the sum of t_A, i_A, and f_A, i.e.

$$0 \leqslant \max t_A(u_i) + \max i_A(u_i) + \max f_A(u_i) \leqslant N_{max} + N_{max} + N_{max} = -3.$$

Also, a neutrosophic set A is understood as a triple $\langle t_A, i_A, f_A \rangle$ and it can be regarded as consisting of hyper-valued or p-adic valued degrees.

As we see, in neutrosophic sets, indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent. This assumption is very important in many applications such as information fusion in which we try to combine the data from different sensors. Neutrosophic sets are proposed for the first time in the framework of neutrosophy that was introduced by Smarandache in 1980: "It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra" [51].

Neutrosophic set is a powerful general formal framework which generalizes the concept of the fuzzy set [58], interval valued fuzzy set [56], intuitionistic fuzzy set, and interval valued intuitionistic fuzzy set.

Suppose that t_A, i_A, f_A are subintervals of ${}^*[0,1]$. Then a neutrosophic set A is called an *interval* one.

When the universe of discourse U is continuous, an interval neutrosophic set A can be written as

$$A = \int_U \langle t_A(u_i), i_A(u_i), f_A(u_i)\rangle / u_i, \quad u_i \in U.$$

When U is discrete, then

$$A = \sum_{i=1}^{n} \langle t_A(u_i), i_A(u_i), f_A(u_i)\rangle / u_i, \quad u_i \in U.$$

The interval neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information which exist in real world. It can be readily seen that the interval neutrosophic set generalizes the following sets:

- the classical set, $i_A = \emptyset$, $\min t_A = \max t_A = 0$ or 1, $\min f_A = \max f_A = 0$ or 1 and $\max t_A + \max f_A = 1$.
- the fuzzy set, $i_A = \emptyset$, $\min t_A = \max t_A \in [0,1]$, $\min f_A = \max f_A \in [0,1]$ and $\max t_A + \max f_A = 1$.
- the interval valued fuzzy set, $i_A = \emptyset$, $\min t_A$, $\max t_A$, $\min f_A$, $\max f_A \in [0,1]$, $\max t_A + \min f_A = 1$ and $\min t_A + \max f_A = 1$.
- the intuitionistic fuzzy set, $i_A = \emptyset$, $\min t_A = \max t_A \in [0,1]$, $\min f_A = \max f_A \in [0,1]$ and $\max t_A + \max f_A \leqslant 1$.
- the interval valued intuitionistic fuzzy set, $i_A = \emptyset$, $\min t_A$, $\max t_A$, $\min f_A$, $\max f_A \in [0,1]$, $\max t_A + \min f_A \leqslant 1$.
- the paraconsistent set, $i_A = \emptyset$, $\min t_A = \max t_A \in [0,1]$, $\min f_A = \max f_A \in [0,1]$ and $\max t_A + \max f_A > 1$.
- the interval valued paraconsistent set, $i_A = \emptyset$, $\min t_A$, $\max t_A$, $\min f_A$, $\max f_A \in [0,1]$, $\max t_A + \min f_A > 1$.

Let S_1 and S_2 be two real standard or non-standard subsets of ${}^*[0,1]$, then $S_1 + S_2 = \{x\colon x = s_1 + s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, ${}^*a + S_2 = \{x\colon x = {}^*a + s_2, s_2 \in S_2\}$, $S_1 - S_2 = \{x\colon x = s_1 - s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $S_1 \cdot S_2 = \{x\colon x = s_1 \cdot s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $\max(S_1, S_2) = \{x\colon x = \max(s_1, s_2), s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $\min(S_1, S_2) = \{x\colon x = \min(s_1, s_2), s_1 \in S_1 \text{ and } s_2 \in S_2\}$.

1. **The complement of a neutrosophic set A is defined as follows**
   - the Łukasiewicz complement:

$$\neg_L A = \langle {}^*1 - t_A, {}^*1 - i_A, {}^*1 - f_A\rangle, \quad t_A, i_A, f_A \subseteq ({}^*[0,1])^U,$$

$$\neg_L A = \langle N_{max} - t_A, N_{max} - i_A, N_{max} - f_A\rangle, \quad t_A, i_A, f_A \subseteq (\mathbf{Z}_p)^U,$$

- the GÖDEL complement:

$$\neg_G A = \langle \neg_G t_A, \neg_G i_A, \neg_G f_A \rangle, \ \ t_A, i_A, f_A \subseteq ({}^*[0,1])^U,$$

$$\neg_G A = \langle \neg_G t_A, \neg_G i_A, \neg_G f_A \rangle, \ \ t_A, i_A, f_A \subseteq (\mathbf{Z}_p)^U,$$

where $\neg_G t_A = \{\neg_G x : x \in t_A\}$, $\neg_G i_A = \{\neg_G x : x \in i_A\}$, $\neg_G f_A = \{\neg_G x \colon x \in f_A\}$,

- the Product complement:

$$\neg_\Pi A = \langle \neg_\Pi t_A, \neg_\Pi i_A, \neg_\Pi f_A \rangle, \ \ t_A, i_A, f_A \subseteq ({}^*[0,1])^U,$$

$$\neg_\Pi A = \langle \neg_\Pi t_A, \neg_\Pi i_A, \neg_\Pi f_A \rangle, \ \ t_A, i_A, f_A \subseteq (\mathbf{Z}_p)^U,$$

where $\neg_\Pi t_A = \{\neg_\Pi x \colon x \in t_A\}$, $\neg_\Pi i_A = \{\neg_\Pi x \colon x \in i_A\}$, $\neg_\Pi f_A = \{\neg_\Pi x \colon x \in f_A\}$.

2. **The implication of two neutrosophic sets A and B is defined as follows**

- the ŁUKASIEWICZ implication:

$$A \rightarrow_L B = \langle {}^*1 - \max(t_A, t_B) + t_B, {}^*1 - \max(i_A, i_B) + i_B, {}^*1 - \max(f_A, f_B) + f_B \rangle, \ \ t_A,\, i_A,\, f_A,\, t_B,\, i_B,\, f_B \subseteq ({}^*[0,1])^U,$$

$$A \rightarrow_L B = \langle N_{max} - \max(t_A, t_B) + t_B, N_{max} - \max(i_A, i_B) + i_B, N_{max} - \max(f_A, f_B) + f_B \rangle, \ \ t_A,\, i_A,\, f_A,\, t_B,\, i_B,\, f_B \subseteq (\mathbf{Z}_p)^U,$$

- the GÖDEL implication:

$$A \rightarrow_G B = \langle t_A \rightarrow_G t_B, i_A \rightarrow_G i_B, f_A \rightarrow_G f_B \rangle, \ \ t_A,\, i_A,\, f_A,\, t_B,\, i_B,\, f_B \subseteq ({}^*[0,1])^U,$$

$$A \rightarrow_G B = \langle t_A \rightarrow_G t_B, i_A \rightarrow_G i_B, f_A \rightarrow_G f_B \rangle, \ \ t_A,\, i_A,\, f_A,\, t_B,\, i_B,\, f_B \subseteq (\mathbf{Z}_p)^U,$$

where $t_A \rightarrow_G t_B = \{x \colon x = s_1 \rightarrow_G s_2, s_1 \in t_A \text{ and } s_2 \in t_B\}$, $i_A \rightarrow_G i_B = \{x \colon x = s_1 \rightarrow_G s_2, s_1 \in i_A \text{ and } s_2 \in i_B\}$, $f_A \rightarrow_G f_B = \{x \colon x = s_1 \rightarrow_G s_2, s_1 \in f_A \text{ and } s_2 \in f_B\}$,

- the Product implication:

$$A \rightarrow_\Pi B = \langle t_A \rightarrow_\Pi t_B, i_A \rightarrow_\Pi i_B, f_A \rightarrow_\Pi f_B \rangle, \ t_A, i_A, f_A, t_B, i_B, f_B \subseteq ({}^*[0,1])^U,$$

$$A \rightarrow_\Pi B = \langle t_A \rightarrow_\Pi t_B, i_A \rightarrow_\Pi i_B, f_A \rightarrow_\Pi f_B \rangle, \ t_A, i_A, f_A, t_B, i_B, f_B \subseteq (\mathbf{Z}_p)^U,$$

where $t_A \to_\Pi t_B = \{x\colon x = s_1 \to_\Pi s_2, s_1 \in t_A \text{ and } s_2 \in t_B\}$, $i_A \to_\Pi i_B = \{x\colon x = s_1 \to_\Pi s_2, s_1 \in i_A \text{ and } s_2 \in i_B\}$, $f_A \to_\Pi f_B = \{x\colon x = s_1 \to_\Pi s_2, s_1 \in f_A \text{ and } s_2 \in f_B\}$,

3. **The intersection of two neutrosophic sets A and B is defined as follows**

   - the Łukasiewicz intersection:

     $A \&_L B = \langle \max(t_A, {}^*1 - t_B) + t_B - {}^*1,\ \max(i_A, {}^*1 - i_B) + i_B - {}^*1,\ \max(f_A, {}^*1 - f_B) + f_B - {}^*1\rangle,\ \ t_A,\, i_A,\, f_A,\, t_B,\, i_B,\, f_B \subseteq ({}^*[0,1])^U$,

     $A \&_L B = \langle \max(t_A, N_{max} - t_B) + t_B - N_{max}, \max(i_A, N_{max} - i_B) + i_B - N_{max}, \max(f_A, N_{max} - f_B) + f_B - N_{max}\rangle,\ \ t_A,\, i_A,\, f_A,\, t_B,\, i_B,\, f_B \subseteq (\mathbf{Z}_p)^U$,

   - the Gödel intersection:

     $A \&_G B = \langle \min(t_A, t_B), \min(i_A, i_B), \min(f_A, f_B)\rangle,\ \ t_A,\, i_A,\, f_A,\, t_B,\, i_B,\, f_B \subseteq ({}^*[0,1])^U$,

     $A \&_G B = \langle \min(t_A, t_B), \min(i_A, i_B), \min(f_A, f_B)\rangle,\ \ t_A,\, i_A,\, f_A,\, t_B,\, i_B,\, f_B \subseteq (\mathbf{Z}_p)^U$,

   - the Product intersection:

     $A \&_\Pi B = \langle t_A \cdot t_B, i_A \cdot i_B, f_A \cdot f_B\rangle,\ \ t_A, i_A, f_A, t_B, i_B, f_B \subseteq ({}^*[0,1])^U$,

     $A \&_\Pi B = \langle t_A \cdot t_B, i_A \cdot i_B, f_A \cdot f_B\rangle,\ \ t_A, i_A, f_A, t_B, i_B, f_B \subseteq (\mathbf{Z}_p)^U$,

Thus, we can extend the logical operations of fuzzy logic to the case of neutrosophic sets.

9 Interval neutrosophic matrix logic

Interval neutrosophic logic proposed in [51], [52] generalizes the interval valued fuzzy logic, the non-Archimedean valued fuzzy logic, and paraconsistent logics. In the interval neutrosophic logic, we consider not only truth-degree and falsity-degree, but also indeterminacy-degree which can reliably capture more information under uncertainty.

Now consider *hyper-valued interval neutrosophic matrix logic* INL defined as the ordered system $\langle({}^*[0,1])^3, \neg_{INL}, \to_{INL}, \vee_{INL}, \wedge_{INL}, \widetilde{\exists}_{INL}, \widetilde{\forall}_{INL}, \{\langle {}^*1, {}^*0, {}^*0\rangle\}\rangle$ where

1. for all $\langle t, i, f\rangle \in ({}^*[0,1])^3$, $\neg_{INL}\langle t, i, f\rangle = \langle f, 1 - i, t\rangle$,
2. for all $\langle t_1, i_1, f_1\rangle, \langle t_2, i_2, f_2\rangle \in ({}^*[0,1])^3$, $\langle t_1, i_1, f_1\rangle \to_{INL} \langle t_2, i_2, f_2\rangle = \langle \min({}^*1, {}^*1 - t_1 + t_2), \max({}^*0, i_2 - i_1), \max({}^*0, f_2 - f_1)\rangle$,

3. for all $\langle t_1,\ i_1,\ f_1\rangle,\ \langle t_2,\ i_2,\ f_2\rangle \in ({}^*[0,1])^3$, $\langle t_1,\ i_1,\ f_1\rangle \wedge_{INL} \langle t_2,\ i_2,\ f_2\rangle = \langle \min(t_1,t_2), \max(i_1,i_2), \max(f_1,f_2)\rangle$,
4. for all $\langle t_1,\ i_1,\ f_1\rangle,\ \langle t_2,\ i_2,\ f_2\rangle \in ({}^*[0,1])^3$, $\langle t_1,\ i_1,\ f_1\rangle \vee_{INL} \langle t_2,\ i_2,\ f_2\rangle = \langle \max(t_1,t_2), \min(i_1,i_2), \min(f_1,f_2)\rangle$,
5. for a subset $\langle M_1,\ M_2,\ M_3\rangle \subseteq ({}^*[0,1])^3$, $\widetilde{\exists}(\langle M_1,\ M_2,\ M_3\rangle) = \langle \max(M_1), \min(M_2), \min(M_3)\rangle$,
6. for a subset $\langle M_1,\ M_2,\ M_3\rangle \subseteq ({}^*[0,1])^3$, $\widetilde{\forall}(\langle M_1,\ M_2,\ M_3\rangle) = \langle \min(M_1), \max(M_2), \max(M_3)\rangle$,
7. $\{\langle {}^*1,\ {}^*0,\ {}^*0\rangle\}$ is the set of designated truth values.

Now consider *p-adic valued interval neutrosophic matrix logic INL* defined as the ordered system $\langle(\mathbf{Z}_p)^3,\ \neg_{INL},\ \rightarrow_{INL},\ \vee_{INL},\ \wedge_{INL},\ \widetilde{\exists}_{INL},\ \widetilde{\forall}_{INL},\ \{\langle N_{max},\ 0,\ 0\rangle\}\rangle$ where

1. for all $\langle t,i,f\rangle \in (\mathbf{Z}_p)^3$, $\neg_{INL}\langle t,i,f\rangle = \langle f, 1-i, t\rangle$,
2. for all $\langle t_1,i_1,f_1\rangle, \langle t_2,i_2,f_2\rangle \in (\mathbf{Z}_p)^3$, $\langle t_1,\ i_1,\ f_1\rangle \rightarrow_{INL} \langle t_2,\ i_2,\ f_2\rangle = \langle N_{max} - \max(t_1,t_2) + t_2,\ \max(0, i_2 - i_1), \max(0, f_2 - f_1)\rangle$,
3. for all $\langle t_1,\ i_1,\ f_1\rangle,\ \langle t_2,\ i_2,\ f_2\rangle \in (\mathbf{Z}_p)^3$, $\langle t_1,\ i_1,\ f_1\rangle \wedge_{INL} \langle t_2,\ i_2,\ f_2\rangle = \langle \min(t_1,t_2), \max(i_1,i_2), \max(f_1,f_2)\rangle$,
4. for all $\langle t_1,\ i_1,\ f_1\rangle,\ \langle t_2,\ i_2,\ f_2\rangle \in (\mathbf{Z}_p)^3$, $\langle t_1,\ i_1,\ f_1\rangle \vee_{INL} \langle t_2,\ i_2,\ f_2\rangle = \langle \max(t_1,t_2), \min(i_1,i_2), \min(f_1,f_2)\rangle$,
5. for a subset $\langle M_1,\ M_2,\ M_3\rangle \subseteq (\mathbf{Z}_p)^3$, $\widetilde{\exists}(\langle M_1,\ M_2,\ M_3\rangle) = \langle \max(M_1), \min(M_2), \min(M_3)\rangle$,
6. for a subset $\langle M_1,\ M_2,\ M_3\rangle \subseteq (\mathbf{Z}_p)^3$, $\widetilde{\forall}(\langle M_1,\ M_2,\ M_3\rangle) = \langle \min(M_1), \max(M_2), \max(M_3)\rangle$,
7. $\{\langle N_{max},\ 0,\ 0\rangle\}$ is the set of designated truth values.

As we see, interval neutrosophic matrix logic INL is an extension of the non-Archimedean valued Łukasiewicz matrix logic.

10 Hilbert's type calculus for interval neutrosophic propositional logic

Interval neutrosophic calculus denoted by **INL** is built in the framework of the language $\mathcal{L}'$, but its semantics is different.

An interpretation is defined in the standard way. Extend the valuation of $\mathcal{L}'$ to the valuation for interval neutrosophic calculus as follows.

Definition 11 *Given an interpretation $I = \langle \mathcal{M}, s\rangle$ and a valuation val_I^{∞} of $\mathcal{L}'$, we define the hyper-valued interval neutrosophic valuation $\mathrm{val}_I^{\infty,INL}(\cdot)$ to be a mapping from formulas of the form φ^{∞} of $\mathcal{L}'$ to interval neutrosophic matrix logic INL as follows:*

$$\mathrm{val}_I^{\infty,INL}(\varphi^{\infty}) = \langle \mathrm{val}_I^{\infty}(\varphi^{\infty}) = t(\varphi^{\infty}), i(\varphi^{\infty}), f(\varphi^{\infty})\rangle \in ({}^*[0,1])^3.$$

Definition 12 *Given an interpretation $I = \langle \mathcal{M}, s\rangle$ and a valuation val_I^∞ of $\mathcal{L}'$, we define the p-adic valued interval neutrosophic valuation $\mathrm{val}_I^{\infty,INL}(\cdot)$ to be a mapping from formulas of the form φ^∞ of $\mathcal{L}'$ to interval neutrosophic matrix logic INL as follows:*

$$\mathrm{val}_I^{\infty,INL}(\varphi^\infty) = \langle \mathrm{val}_I^\infty(\varphi^\infty) = t(\varphi^\infty), i(\varphi^\infty), f(\varphi^\infty)\rangle \in (\mathbf{Z}_p)^3.$$

We say that an INL-structure $\mathcal{M}$ is a *model* of an **INL**-theory T iff

$$\mathrm{val}_I^{\infty,INL}(\varphi^\infty) = \langle {}^*1, {}^*0, {}^*0\rangle$$

on $\mathcal{M}$ for each $\varphi^\infty \in T$.

Proposition 1 *In the matrix logic INL, modus ponens is preserved, i.e. if φ^∞ and $\varphi^\infty \rightarrow_{INL} \psi^\infty$ are INL-tautologies, then ψ^∞ is also an INL-tautology.*

Proof. Consider the hyper-valued case. Since φ^∞ and $\varphi^\infty \rightarrow_{INL} \psi^\infty$ are INL-tautologies, then

$$\mathrm{val}_I^{\infty,INL}(\varphi) = \mathrm{val}_I^{\infty,INL}(\varphi^\infty \rightarrow_{INL} \psi^\infty) = \langle {}^*1, {}^*0, {}^*0\rangle,$$

that is $\mathrm{val}_I^{\infty,INL}(\varphi^\infty) = \langle {}^*1, {}^*0, {}^*0\rangle$, $\mathrm{val}_I^{\infty,INL}(\varphi^\infty \rightarrow_{INL} \psi^\infty) = \langle \min({}^*1, {}^*1 - t(\varphi^\infty) + t(\psi)) = {}^*1, \max({}^*0, i(\psi^\infty) - i(\varphi^\infty)) = {}^*0, \max({}^*0, f(\psi^\infty) - f(\varphi^\infty)) = {}^*0\rangle$. Hence, $t(\psi^\infty) = {}^*1$, $i(\psi^\infty) = f(\psi^\infty) = {}^*0$. So ψ^∞ is an INL-tautology. □

The following axiom schemata for **INL** were regarded in [57].

$$\psi^\infty \rightarrow_{INL} (\varphi^\infty \rightarrow_{INL} \psi^\infty), \tag{23}$$

$$(\psi^\infty \wedge_{INL} \varphi^\infty) \rightarrow_{INL} \varphi^\infty, \tag{24}$$

$$\psi^\infty \rightarrow_{INL} (\psi^\infty \vee_{INL} \varphi^\infty), \tag{25}$$

$$\psi^\infty \rightarrow_{INL} (\varphi^\infty \rightarrow_{INL} (\psi^\infty \wedge_{INL} \varphi^\infty)), \tag{26}$$

$$\begin{split}(\psi^\infty \rightarrow_{INL} \chi^\infty) \rightarrow_{INL} ((\varphi^\infty \rightarrow_{INL} \chi^\infty) \rightarrow_{INL} \\ ((\psi^\infty \vee_{INL} \varphi^\infty) \rightarrow_{INL} \chi^\infty)),\end{split} \tag{27}$$

$$\begin{split}((\psi^\infty \vee_{INL} \varphi^\infty) \rightarrow_{INL} \chi^\infty) \leftrightarrow_{INL} \\ ((\psi^\infty \rightarrow_{INL} \chi^\infty) \wedge_{INL} (\varphi^\infty \rightarrow_{INL} \chi^\infty)),\end{split} \tag{28}$$

$$(\psi^\infty \rightarrow_{INL} \varphi^\infty) \leftrightarrow_{INL} (\neg_{INL}\varphi^\infty \rightarrow_{INL} \neg_{INL}\psi^\infty), \tag{29}$$

$$(\psi^\infty \rightarrow_{INL} \varphi^\infty) \wedge_{INL} (\varphi^\infty \rightarrow_{INL} \chi^\infty) \rightarrow_{INL} (\psi^\infty \rightarrow_{INL} \chi^\infty), \tag{30}$$

$$
\begin{aligned}
(\psi^\infty \rightarrow_{INL} \varphi^\infty) \leftrightarrow_{INL} \\
(\psi^\infty \leftrightarrow_{INL} (\psi^\infty \wedge_{INL} \varphi^\infty)) \leftrightarrow_{INL} \\
(\varphi^\infty \rightarrow_{INL} (\psi^\infty \vee_{INL} \varphi^\infty)).
\end{aligned} \tag{31}
$$

The only inference rule of **INL** is modus ponens.

We can take also the non-Archimedean case of axiom schemata for the axiomatization of **INL**, because **INL** is a generalization of non-Archimedean valued ŁUKASIEWICZ's logic (see the previous section). This means that we can also set **INL** as generalization of non-Archimedean valued GÖDEL's or Product logics.

11 Conclusion

The informal sense of ARCHIMEDES' axiom is that anything can be measured by a ruler. The negation of this axiom allows to postulate infinitesimals and infinitely large integers and, as a result, to consider non-wellfounded and neutrality phenomena. In this book we examine the non-Archimedean fuzziness, i.e. fuzziness that runs over the non-Archimedean number systems. We show that this fuzziness is constructed in the framework of the t-norm based approach. We consider two cases of the non-Archimedean fuzziness: one with many-validity in the interval $[0, 1]$ of hypernumbers and one with many-validity in the ring $\mathbf{Z}_p$ of p-adic integers.

Hyper-valued (resp. p-adic valued) interval neutrosophic logic **INL** by which we can describe neutrality phenomena is an extension of non-Archimedean valued fuzzy logic that is obtained by adding a truth triple $\langle t, i, f \rangle \in ({}^*[0, 1])^3$ (resp. $\langle t, i, f \rangle \in (\mathbf{Z}_p)^3$) instead of one truth value $t \in {}^*[0, 1]$ (resp. $t \in \mathbf{Z}_p$) to the formula valuation, where t is a truth-degree, i is an indeterminacy-degree, and f is a falsity-degree.

References

[1] Baaz, M., Fermüller, C. G., Intuitionistic Counterparts of Finite-valued Logics. *Logic Colloquium '95 (Abstracts of Contributed Papers)*, Haifa, August 1995.

[2] Bachman, G., *Introduction to p-adic numbers and valuation theory*, Academic Press, 1964.

[3] Běhounek, L., Cintula, P., Fuzzy class theory, *Fuzzy Sets and Systems*, 154 (1):34-55, 2005.

[4] Běhounek, L., Cintula, P., General logical formalism for fuzzy mathematics: Methodology and apparatus. *Fuzzy Logic, Soft Computing and Computational Intelligence: Eleventh International Fuzzy Systems Association World Congress*. Tsinghua University Press Springer, 2:1227-1232, 2005.

[5] Bolc, L., Borowik, P., *Many-Valued Logics. Theoretical Foundations*. Springer, Berlin, 1992.

[6] Cignoli, R., D'Ottaviano, I. M. L., Mundici, D., *Algebraic Foundations of Many-Valued Reasoning*, vol. 7 of Trends in Logic. Kluwer, Dordrecht, 1999.

[7] Cignoli, R., Esteva, F., Godo, L., Torrens, A., Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing*, 4(2):106-112, 2000.

[8] Cintula, P., Advances in the ŁΠ and ŁΠ$\frac{1}{2}$ logics. *Archive for Mathematical Logic*, 42 (5):449-468, 2003.

[9] Cintula, P., The ŁΠ and ŁΠ$\frac{1}{2}$ propositional and predicate logics, *Fuzzy Sets and Systems*, 124 (3):21-34, 2001.

[10] Cutland, N. J. (edit.), *Nonstandard Analysis and its Applications*. Cambridge University Press, Cambridge, 1988.

[11] Davis, M., *Applied Nonstandard Analysis*, John Wiley and Sons, New York, 1977.

[12] Dezert, J., Smarandache, F., On the generation of hyper-power sets, *Proc. of Fusion 2003, Cairns, Australia, July 8–11*, 2003.

[13] Dezert, J., Smarandache, F., Partial ordering of hyper-power sets and matrix representation of belief functions within DSmT, *Proc. of the 6th Int. Conf. on inf. fusion (Fusion 2003), Cairns, Australia, July 8–11*, 2003.

[14] Epstein, G., *Multiple-valued Logic Design*. Institute of Physics Publishing, Bristol, 1993.

[15] Esteva, F., Godo, L., Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124:271-288, 2001.

[16] Esteva, F., Godo, L., Garcia-Cerdana, A., On the hierarchy of t-norm based residuated fuzzy logics. Fitting, M., Orlowska, E. (edits.), *Beyond two: theory and applications of multiple-valued logic*, Physica-Verlag, 251-272, 2003.

[17] Gabbay, D. M., Guenther, F. (edits.), *Handbook of Philosophical Logic*, Reidel, Dordrecht, 1986.

[18] Gentzen, G., Untersuchungen über das logische Schließen I–II. *Math. Z.*, 39:176–210, 405-431, 1934.

[19] Gödel, K., Zum intuitionistischen Aussagenkalkül. *Anz. Akad. Wiss. Wien*, 69:65–66, 1932.

[20] Gottwald, S., *Mehrwertige Logik*. Akademie-Verlag, Berlin, 1989.

[21] Gottwald, S., *A Treatise on Many-Valued Logics*, volume 9 of Studies in Logic and Computation. Research Studies Press, Baldock, 2000.

[22] Gottwald, S., *Fuzzy Sets and Fuzzy Logic: Foundations of Application – from A Mathematical Point of View*. Vieweg, Wiesbaden, 1993.

[23] Gottwald, S., Set theory for fuzzy sets of higher level, *Fuzzy Sets and Systems*, 2:25–51, 1979.

[24] Hähnle, R., *Automated Deduction in Multiple-Valued Logics*. Oxford University Press, 1993.

[25] Hájek, P., *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.

[26] Höhle, U., Klement, E. P. (edits.), *Non-Classical Logics and their Applications to Fuzzy Subsets*, Kluwer, Dordrecht, 1995.

[27] Hurd, A., Loeb, P. A., *An Introduction to Nonstandard Real Analysis*, Academic Press, New York.

[28] Karpenko, A., Characterization of prime numbers in Łukasiewicz logical matrix, *Studia Logica*. Vol. 48, 4:465–478, 1994.

[29] Karpenko, A., The class of precomplete Łukasiewicz's many-volued logics and the law of prime number generation, *Bulletin of the Section of Logic*. Vol. 25, 1:52–57, 1996.

[30] Khrennikov, A. Yu., *p-adic valued distributions in mathematical physics*, Kluwer Academic Publishers, Dordrecht, 1994.

[31] Khrennikov, A. Yu., *Interpretations of Probability*, VSP Int. Sc. Publishers, Utrecht/Tokyo, 1999.

[32] Khrennikov, A. Yu., Schumann, A., Logical Approach to p-adic Probabilities, *Bulletin of the Section of Logic*, 35/1:49–57 2006.

[33] Kirin, V. G., Gentzen's method for the many-valued propositional calculi. *Z. Math. Logik Grundlag. Math.*, 12:317–332, 1966.

[34] Koblitz, N., *p-adic numbers, p-adic analysis and zeta functions*, second edition, Springer-Verlag, 1984.

[35] Łukasiewicz, J., O logice tròjwartościowej, *Ruch Filozoficzny*, 5:169–171, 1920.

[36] Mahler, K., *Introduction to p-adic numbers and their functions*, Second edition, Cambridge University Press, 1981.

[37] Malinowski, G., *Many-valued Logics*, Oxford Logic Guides 25, Oxford University Press, 1993.

[38] Metcalfe, G., Proof Theory for Propositional Fuzzy Logics. PhD thesis, Kings College London, 2003.

[39] Metcalfe, G., Olivetti, N., Gabbay, D., Sequent and hypersequent calculi for abelian and Łukasiewicz logics, *ACM Trans. Comput. Log.* 6, 3. 2005. P. 578 – 613.

[40] Novák, V., On fuzzy type theory, *Fuzzy Sets and Systems*, 149:235–273, 2004.

[41] Novák, V., Perfilieva I., Močkoř J., *Mathematical Principles of Fuzzy Logic*. Kluwer, Boston Dordrecht, 1999.

[42] Rescher, N., *Many-valued Logic*. McGraw-Hill, New York, 1969.

[43] Robert, A. M., *A course in p-adic analysis*, Springer-Verlag, 2000.

[44] Robinson, A., *Non-Standard Analysis*, North-Holland Publ. Co., 1966.

[45] Schumann, A., Non-Archimedean Fuzzy Reasoning, *Fuzzy Systems and Knowledge Discovery (FSKD'07)*, IEEE Press, 2007.

[46] Schumann, A., Non-Archimedean Valued Predicate Logic, *Bulletin of the Section of Logic*, 36/1, 2007.

[47] Schumann, A., Non-Archimedean Valued Sequent Logic, *Eighth International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC'06)*, IEEE Press, 89–92, 2006.

[48] Schumann, A., DSm Models and non-Archimedean Reasoning, Smarandache, F., Dezert, J. (edits.), *Advances and Applications of DSmT (Collected works)*, Vol. 2, American Research Press, Rehoboth, 183–204, 2006.

[49] Schumann, A., p-Adic Multiple-Validity and p-Adic Valued Logical Calculi, *Journal of Multiple-Valued Logic and Soft Computing*, 13 (1–2):29–60, 2007.

[50] Shafer, G., *A Mathematical Theory of Evidence*, Princeton Univ. Press, Princeton, NJ, 1976.

[51] Smarandache, F., *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Probability, and Statistics*, Amer. Research Press, Rehoboth, 2000.

[52] Smarandache, F., A Unifying Field in Logics: Neutrosophic Logic, *Multiple-valued logic, An international journal*, Vol. 8, 3:385–438, 2002.

[53] Smarandache, F., Neutrosophy: A new branch of philosophy, *Multiple-valued logic, An international journal*, Vol. 8, 3:297–384, 2002.

[54] Smarandache, F., Dezert, J. (edits.), *Applications and Advances of DSmT for Information Fusion*, Collected Works, American Research Press, Rehoboth, June 2004,

[55] Smarandache, F., Dezert, J. (edits.), *Advances and Applications of DSmT (Collected works)*, Vol. 2, American Research Press, Rehoboth, 2006.

[56] Turksen, I., Interval valued fuzzy sets based on normal forms, *Fuzzy Sets and Systems* 20:191–210, 1986.

[57] Wang, W., Smarandache, F., Zhang, Y.-Q., Sunderraman, R., *Interval Neutrosophic Sets and Logic: Theory and Aplications in Computing*, Hexis, Arizona, 2005.

[58] Zadeh, L., Fuzzy sets, *Inform. and Control* 8:338–353, 1965.

archimedes axiom non-archimedean structure neutrosophic logic neutrality phenomenon special format general way standard many-valued logic ring zp non-archimedean many-valued logic non-archimedean multi-valued logic different extension informal sense appropriate rejection truth value large family interval neutrosophic logic p-adic integer analytic calculus logical multiple-validity.