RESEARCH PROJECT

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1. Quantization of singular spaces

The first goal of my project consists in quantizing singular spaces, more precisely in extending the results about equivariant quantization on orbifolds established in [16] to stratified spaces (see [15]). Such spaces have a great importance in Physics: indeed, in many problems related to Mechanics, we are often brought to consider the quotient of the configuration space by the action of a symmetry group. This quotient has generally singularities: it can be often identified with an orbifold or a stratified space ([15], [14]).

2. Quantization of supermanifolds

Another direction of my research consists in the quest of invariant quantizations on supermanifolds. The interest for the supermanifolds increased considerably these last years because of their presence in supersymmetry theories (see [4], [3], [18]...).

A first step in the quantization of supermanifolds is the study of the different types of invariance which one can impose on the quantization in the "flat" case, i.e. when the quantization is defined on the superspace $\mathbb{R}^{p|q}$ and when one imposes on it to commute with the Lie derivative in the direction of some vector fields. In classical geometry, the authors of [2] showed that the maximal subalgebras in the Lie algebra of vector fields on $\mathbb{R}^m$ can be built from graded Lie algebras called "IFFT-algebras". These algebras have been classified by Kobayashi and Nagano in [9]. In [1], the existence of quantizations which are equivariant with respect to the action of the IFFT-algebras was proved. We can then wonder if there exists in supergeometry a result corresponding to the classification of IFFT-algebras and if it is possible to prove in a generic way the existence of quantizations that are equivariant with respect to the action of the maximal subalgebras in the Lie algebra of vector fields on $\mathbb{R}^{p|q}$.

In [7], we prove the existence of $\mathfrak{osp}(p+1, q+1|2r)$-equivariant quantizations for differential operators acting between λ and μ-densities when the superdimension $p+q-2r$ is different from 0 and when $\delta = \mu - \lambda$ is different from some values called resonant values. It remains to determine what happens when δ is resonant. When the superdimension vanishes, we proved the existence of equivariant quantizations at the order two; their existence at an arbitrary order is still a conjecture.

In [12], I proved with N. Mellouli and A. Nibirantiza the existence and the uniqueness of an $\mathfrak{spo}(2|2)$-equivariant quantization on the supercircle $S^{1|2}$ for differential operators acting between λ- and μ-densities when $\delta = \mu - \lambda$ is different from some non-resonant values. After this work, some questions remain unresolved: generalization of this result for arbitrary dimensions, analysis of the resonant situations, study of the problem when the filtration on the space of differential operators is the classical one. The algebra $\mathfrak{spo}(2|2)$ being a subalgebra of the algebra $\mathfrak{sl}(2|2)$, we can also wonder if there is a link between the $\mathfrak{sl}(2|2)$-equivariant quantization built in [11] and the $\mathfrak{spo}(2|2)$-equivariant quantization constructed in [12].

In [8], we prove the existence of a natural and projectively invariant quantization (a quantization which depends in a natural way on a projective class of connections) on supermanifolds for differential operators acting between densities. This problem of quantization was solved on a supermanifold of dimension $(n|m)$ with $n - m \neq -1$. For this particular superdimension, the resolution proposed in [8] does not work because the Thomas connection, which is the main tool of the construction, is not defined in this case. One may then wonder whether it is possible to solve the problem of quantization in this particular situation by another method.

In order to solve the problem of the natural and projectively invariant quantization for differential operators acting between sections of other natural fiber bundles or in order to solve the problem of the natural and conformally invariant quantization (a quantization which depends in a natural way on a conformal class of metrics), it could be necessary to transpose in supergeometry the theory of the Cartan fiber bundles and connections used in [10]. To this aim, the reference [17] is certainly useful: indeed, the author develops in this book the theory of principal and associated fiber bundles and the theory of connections.

The equivariant quantizations allow to build invariant star-products on the cotangent bundle of a manifold. In [5], the authors prove the existence of $\mathrm{SL}(m+1,\mathbb{R})$- (resp. $\mathrm{SO}(p+1,q+1)$)-invariant star-products on the cotangent bundle of a projectively (resp. conformally) flat manifold. One can wonder if it is possible to extend this result in the framework of supergeometry, by defining first the notion of star-product on the cotangent bundle over a supermanifold and by using in a second step the $\mathfrak{sl}(p+1|q)$- and $\mathfrak{osp}(p+1,q+1|2r)$-equivariant quantizations which were built respectively in [11] and [7]. We can remark that the notion of cotangent bundle over a supermanifold is already defined e.g. in [17].

3. Symmetries of the Laplacian

As explained in the review on my current research, I describe in [13] with J.-P. Michel and J. Silhan the structure of second-order (conformal) symmetries of the Yamabe Laplacian Δ_Y on a pseudo-Riemannian manifold (M,g). A necessary and sufficient condition to have the existence of a (conformal) symmetry of Δ_Y is the existence of a (conformal) Killing tensor of degree 2 satisfying an additional property. The next step in the study of the (conformal) symmetries of Δ_Y is the study of the (conformal) symmetries of Δ_Y at an arbitrary order.

In order to achieve this task, we should in a first step compute the obstructions to the existence of (conformal) symmetries. In a second step, using these obstructions, we should analyze as in [13] the necessary and sufficient conditions under which the quantization of a symbol is a (conformal) symmetry of Δ_Y.

In [13], we prove that a necessary and sufficient condition for the R-separation in an orthogonal coordinates system of the Helmholtz equation

$$(\Delta_Y + V)\psi = E\psi,$$

where V is a fixed potential and where $E \in \mathbb{R}$ is a free parameter, is linked to the existence of second-order symmetries of $\Delta_Y + V$.

In [6], the author shows that, if the Hamilton-Jacobi equation

$$\mathrm{g}^{ij}(\partial_i\psi)(\partial_j\psi) = E$$

separates in a coordinate system (x^i) (it means that there exists a solution of this equation that reads in this coordinate system as $\prod_{i=1}^n f_i(x^i)$), then the usual Helmholtz equation

$$\Delta\psi = E\psi,$$

where Δ denotes the Laplace-Beltrami operator, separates in the same coordinate system if and only if the Ricci tensor associated with g vanishes (this last condition is called the Robertson's condition).

It would be interesting to find a geometrical condition analogous to the Robertson's condition for the R-separation of the Helmholtz equation. Actually, we have already sufficient conditions that ensure the R-separation of the Helmholtz equation in terms of the Weyl or the Cotton-York tensor, but these conditions are not necessary.

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