A Vector Space Approach to Models and Optimization

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To

Patricia
Scott
Brett
Jonathan
Jennifer
Christopher

Every important idea is simple.

War and Peace
Count Leo Tolstoy

SYSTEMS ENGINEERING AND ANALYSIS SERIES

In a society which is producing more people, more materials, more things, and more information than ever before, systems engineering is indispensable in meeting the challenge of complexity. This series of books is an attempt to bring together in a complementary as well as unified fashion the many specialties of the subject, such as modeling and simulation, computing, control, probability and statistics, optimization, reliability, and economics, and to emphasize the interrelationship between them.

The aim is to make the series as comprehensive as possible without dwelling on the myriad details of each specialty and at the same time to provide a broad basic framework on which to build these details. The design of these books will be fundamental in nature to meet the needs of students and engineers and to insure they remain of lasting interest and importance.

Preface

Models and optimization are fundamental to the design and operation of complex systems. This book is intended as an intuitive, probing, unified treatment of the mathematics of model analysis and optimization. It explores in a unifying framework the structure of deterministic linear system models and the optimization of both linear and nonlinear models. The unification is accomplished by means of the vector space language and a relatively small number of vector space concepts. The mathematical concepts and techniques, although not new, become more accessible when treated in an intuitive, unified manner.

This book is broader in coverage than most books on the subject, and is relatively low in its level of mathematical sophistication. I have de-emphasized mathematical proofs; I have attempted instead to develop concepts by means of geometrical intuition and analogies to ideas familiar to engineering graduates. All concepts are illustrated with specific detailed examples. In addition, I have tried to relate the mathematical concepts to the real world by presenting practical applications and by discussing practical computer implementations of techniques for model analysis and optimization. Thus the development is less sterile than the treatments found in mathematics books.

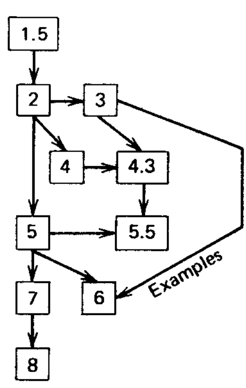
I have attempted to build up the mathematical machinery in a way that demonstrates what can and cannot be accomplished with each tool. This methodical buildup helps to develop a fundamental feel for the mathematical concepts. For example, I withhold the definition of the inner product until late in the development in order that it be clear that perpendicular coordinate systems are not fundamental to the modeling process.

The background required of the reader is a familiarity with elementary matrix manipulations and elementary differential equation concepts. The selection of topics and the order of presentation reflect seven years of experience in presenting the material to full-time graduate students and to practicing engineers at the Moore School of Electrical Engineering, University of Pennsylvania. The book is designed for use as a text in a two-semester course sequence for first-year graduate students in engineering, operations research, and other disciplines which deal with systems. As a consequence of the extensive cross-referencing and the numerous detailed examples, the book is also suitable for self-study. At the end of each chapter references that are good general references for much of the material in that chapter are indicated with asterisks (*). Answers to selected problems are included.

The symbols P & C appear frequently throughout the text in reference to the Problems and Comments sections at the end of each chapter. These problems and comments form an important part of the book. Those problems that are in the form of statements are intended to be proved or verified by simple examples. The problems marked with asterisks (*) present concepts which are used later in the book, or which are significant extensions of the text material; these problems should at least be read and understood.

The reader will find that abstract symbols can be understood more easily if they are thought of in terms of simple examples. If possible, concepts should be illustrated geometrically with two- or three-dimensional arrow vectors.

In a two-semester course sequence, it would be appropriate to treat Sections 1.1-5.3 in the first semester (a vector space approach to models) and Sections 5.4-8.5 in the second (a vector space approach to optimization). There is not sufficient time in two semesters to include all the applications if all the mathematical concepts are covered. (I have usually omitted some of Section 4.4 and some of the applications.) By deleting Chapter 3 and by de-emphasizing differential systems and nondiagonalizable matrices in the remaining chapters, the two semesters can be reduced to two quarters. The accompanying diagram shows how the chapters depend on each other.



Because the concepts treated in this book find application in many fields, it is difficult to avoid conflicts between different standards in notation. I have tried to be as consistent as possible with previous standards. Because instructors cannot use bold print at the blackboard, I have avoided the use of boldfaced type as a ***primary*** means of distinguishing vectors and transformations. However, I do use boldfaced type redundantly to ***emphasize*** the interpretation of an object as a vector or transformation of vectors.

I wish to express my appreciation to H. R. Howland, W. A. Gruver, and C. N. Campopiano, who read the full manuscript and suggested helpful improvements. Thanks are also due to Renate Schulz for her help in proofreading and drawing, to Pam Dorny and Nancy Maguire who did most of the typing, and to the Moore School of Electrical Engineering of the University of Pennsylvania which provided support for much of the effort. Most of all, I wish to express my gratitude to my wife and children who waited patiently for the long nights, weekends, and summers to end.

C. Nelson Dorny

May 1975
Philadelphia, Pennsylvania

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Symbols

Scalars

$a, b, c, d, s, t,$ $\alpha, \beta, \gamma, \epsilon, \sigma, \tau$	general scalars	36
$\bar{a}$	complex conjugate of a	239
i, j, k, r	integer subscripts	37
l, m, n, p, q	positive integers	184
ξ_i, η_i	elements of vectors	37
λ_i	eigenvalues	151
λ_i, μ_i, ν_i	Lagrange multipliers	413

Vectors

$\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$	general vectors	36
$\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \boldsymbol{\phi}$	general functions	40
$\boldsymbol{\theta}$	zero vector or zero function	36
1	unit function	455
$\boldsymbol{\varepsilon}_i$	standard basis vectors for $\mathcal{R}^n$ or $\mathcal{M}^{n\times 1}$	48
$\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}$	Lagrange multiplier vectors	415

Vector Spaces

$\mathcal{V}, \mathcal{W}, \mathcal{U}$	general spaces or subspaces	36
$\mathcal{H}$	Hilbert space	276
$\mathcal{R}^n$	real n-tuple space	39
$\mathcal{P}^n$	polynomial functions of order less than n	40
$\mathcal{C}(a,b)$	functions continuous on $[a,b]$	42
$\mathcal{C}^n(a,b)$	functions with continuous nth derivatives	130, 95
ℓ_2	square-summable infinite sequences	38, 276
$\mathcal{L}_2(a,b)$	functions square-integrable on $[a,b]$	42
$\mathcal{M}^{n\times 1}$	$n\times 1$ matrices	39

Ordered Bases

$\mathcal{X}=\{\mathbf{x}_i\}$, $\mathcal{Y}=\{\mathbf{y}_i\}$, $\mathcal{Z}=\{\mathbf{z}_i\}$	general bases	48
$\mathcal{F}=\{\mathbf{f}_i\}$, $\mathcal{G}=\{\mathbf{g}_i\}$, $\mathcal{H}=\{\mathbf{h}_i\}$	function space bases	50
$\mathcal{E}=\{\boldsymbol{\varepsilon}_i\}$	standard basis for $\mathfrak{R}^n$ or $\mathfrak{M}^{n\times 1}$	48
$\mathfrak{N}$	natural basis	49

Transformations

S, T, U, F, G, H	general transformations	56
F, **G**	usually nonlinear transformations	400
F	usually a functional	403
B	bounded linear functional	286
$\boldsymbol{\Theta}$	zero transformation	59
I	identity transformation	58
D	differentiation operator	64
L	general differential operator	103, 64
$\boldsymbol{\nabla}^2$	Laplacian operator	85
$\mathfrak{L}$	Lagrangian functional	420
E	expected value operator	189
$\boldsymbol{\Phi}_i$	penalty function	538
$\boldsymbol{\psi}_c$	penalized objective function	538
$\boldsymbol{\beta}_i$	boundary condition	104
$\boldsymbol{\mathfrak{L}}$	Laplace transform	65

Matrices

A, B, C	general matrices	559, 62
Q, **R**	positive definite matrices	319
S	change-of-coordinates matrix	77
$\mathbf{E}_{ij}$	constituent matrices	212
$\boldsymbol{\Theta}$	zero matrix	559
I	identity matrix	561
$\boldsymbol{\Lambda}$	diagonal or Jordan form	156, 188

Miscellaneous Symbols

$\mathcal{S}$	general set	56
$\mathcal{S}^{\perp}$	orthogonal complement of $\mathcal{S}$	250
$\mathfrak{N}_g$, $\mathfrak{R}_g$	generalized nullspace and range	182
$\triangleq$	"defined as"	18, 37
$\Rightarrow$	"implies"	47

$[a,b]$	real numbers between and including the end points a and b	42
(a,b)	real numbers between but excluding the end points a and b	115
$\langle\cdot,\cdot\rangle$	inner product	239
$\cdot\rangle\langle\cdot$	outer product	326
$\lVert\cdot\rVert$	norm	240, 464, 325
$\times$	Cartesian product	39
$\{\mathbf{x}_i\}$	a set of vectors denoted $\mathbf{x}_1$, $\mathbf{x}_2$, and so on	47
$\mathbf{A}^{\mathrm{T}}$	transpose of matrix **A**	560
$\mathbf{A}^{-1}$, $\mathbf{T}^{-1}$	inverse of **A** or **T**	58, 564, 22
$\bar{\mathbf{A}}$	complex conjugate of **A**	245
$\mathbf{A}^{\dagger}$, $\mathbf{T}^{\dagger}$	pseudoinverse of **A** or **T**	368
$\mathbf{T}^{*}$	adjoint of **T**	288
$(\mathbf{A} \vdots \mathbf{B})$	**A** augmented with **B**	20
$\det(\mathbf{A})$	determinant of **A**	561
$\dim(\mathcal{V})$	dimension of $\mathcal{V}$	53
$\oplus$	direct sum	145
$\overset{\perp}{\oplus}$	orthogonal direct sum	294
$\nabla\mathbf{F}$	gradient of **F**	404
$\mathbf{dG}(\mathbf{x},\mathbf{h})$	Fréchet differential of **G** at **x**	400
$\mathbf{G}'(\mathbf{x})$	Fréchet derivative of **G** at **x**	400
$\mathbf{F}''$	Second Fréchet derivative of **F**	465
$\dfrac{\partial\mathbf{G}}{\partial\mathbf{x}}$	Jacobian matrix of **G**	405
$[\mathbf{x}]_{\mathcal{X}}$	coordinate matrix of x relative to $\mathcal{X}$	49
$[\mathbf{T}]_{\mathcal{X}\mathcal{Y}}$	matrix of **T** relative to $\mathcal{X}$ and $\mathcal{Y}$	72
$\mathbf{Q}_{\mathcal{X}}$	matrix of an inner product relative to $\mathcal{X}$	245
$k(t,s)$	Green's function	110
$\rho_j(t)$	boundary kernel	110
δ	delta function	568, 575

Image space analysis and separationâ€™, Vector Variational Inequalities and Vector Equilibria. Mathematical Theories, Kluwer Acad. Publ., 1999, 153â€“215.Google Scholar. Giannessi, F., and RapcsÃ¡k, T.: â€˜Images, separation of sets and extremum problemsâ€™, in R. P. Agarwal, (ed.): Recent Trends in Optimization Theory and Applications, Ser. Appl. Anal., World Sci., 1995, 79â€“106.Google Scholar.Â How to cite. Cite this entry as: Giannessi F. (2001) Theorems of the alternative and optimization; Vector optimization IMAGE SPACE APPROACH TO OPTIMIZATION. In: Encyclopedia of Optimization. Springer, Boston, MA.