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Mathematics education in the Netherlands: A guided tour[1]

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1 Introduction

This paper addresses mathematics education in the Netherlands and provides a guided tour through the main aspects of the Dutch approach to mathematics education. The title of the paper also refers to the guidance aspects of mathematics education, the role of the teacher and of the curriculum. The tour will focus on the number strand in primary school mathematics. The two main questions to be dealt with are:

1. How is arithmetic taught in primary school in the Netherlands?
2. What contains the Dutch arithmetic curriculum?

Figure 1: The cover of the 'How the Dutch do arithmetic' report

About fifteen years ago, the first question was also investigated in a study called 'How the Dutch do arithmetic'[2] (see Figure 1) (Van den Heuvel-Panhuizen and Goffree, 1986).

[1] An earlier version of this paper was presented at the Research Conference on 'Teaching Arithmetic in England and the Netherlands' (Homerton College, University of Cambridge, 26-27 March 1999). A shortened version of this paper is published in Anghileri (2001); see Van den Heuvel-Panhuizen (2001).

Compared to the prevailing approach to educational research in those days, this was a fairly unconventional study. The teachers were asked to describe their mathematics teaching on one particular day and their written reports had a free format. The teachers could address whatever they thought important with regard to their mathematics teaching on that day. Each teacher was also free to choose the topic for the mathematics lessons on that day.[3] In total 160 reports were analyzed. The teachers' reports gave a thorough overview of classroom practice. The way in which the research findings were reported was also unconventional: the results were presented by means of classroom vignettes. Examples from the teachers' reports were used to provide an image of mathematics teaching in practice. Moreover, annotations to the research findings were put in the margins of each page. Comments, explanations and alternatives were given to the events in the classrooms. The annotations aimed to stimulate readers to reflect on their own teaching and on mathematics education in general.

The Dutch landscape of mathematics education has changed significantly compared to fifteen years ago[4] and the findings of that study are no longer valid. Nonetheless, there are good reasons for mentioning this study. Firstly, it was an effective investigation that could be carried out easily and it produced usable practical information. Such studies can offer a good overview of what is going on in classrooms. Secondly, teachers' annotated reports can be useful in the reform process and in the implementation of a new approach. They can elicit reflection and discussion on education at a practical and a theoretical level.

The main reason for referring to this study, however, is the 'warning' it provided. The analysis of the data showed a broad diversity in classroom practices. In addition, the analysis also revealed a discrepancy between the ideas about teaching methods on paper — the teaching theory so to speak — and what was happening in the classroom, or at least what was reported to be happening in the classroom.[5] This evidence must be kept in mind during the guided tour, which will not take the readers into classrooms and provide them with a sample of Dutch classroom practice, but rather introduce them to the theoretical framework of teaching mathematics and the teaching activities in tune with these ideas.

Of course, this guided tour cannot offer a thorough overview of the Dutch approach to mathematics education. It is too complex and moreover — and this might be a surprise — the

2 In Dutch the title is: 'Zo rekent Nederland.'

3 Of course, the teachers will show themselves in the best light. This, however, is usually forbidden in educational research. In general, the sample of classroom activities should be representative; but can this ever be achieved? Instead of striving for this, the study accurately mirrored what is attainable in classroom practice from the perspective of the teachers. By taking this point of view, the investigation showed where possible improvements might start.

4 Compared to the present, there was greater emphasis on links to reality fifteen years ago. In general, the focus was more on horizontal mathematization than on vertical mathematization.

5 The MORE study also revealed this discrepancy (see Gravemeijer, Van den Heuvel-Panhuizen, Van Donselaar, Ruesink, Streefland, Vermeulen, Te Woerd, and Van der Ploeg, 1993).

difficulty is that there is no unified Dutch approach. Instead, there are some common ideas about the basic what-and-how of teaching mathematics. These ideas have been developed over the past thirty years and the accumulation and repeated revision of these ideas has resulted in what is now called 'Realistic Mathematics Education' (RME). During this period, emphasis has been placed on differing aspects[6] of the theoretical framework guiding Dutch research and developmental work in the field of mathematics education.[7] Along with this diversity, the theoretical framework itself is subject to a constant process of renewal. Inherent to RME, with its basic idea of mathematics as a human activity, is the concept that it can never be considered as a fixed or finished theory of mathematics education. RME is seen as 'work in progress' (Van den Heuvel-Panhuizen, 1998). The different accentuations are the impetus for this continuing development.

Apart from these national activities, much was also learned from what was happening abroad. RME is not an isolated development and has a lot in common with other reform movements in mathematics. This means that in RME you will certainly recognize similarities with your own ideas on teaching and learning mathematics. There may also be some dissimilarities. Again, it is worthwhile reflecting on these differences to search for clues for further improvement in the what-and-how of mathematics education.

2 RME: The Dutch approach to mathematics education

History and basic philosophy

The development of what is now known as RME started around 1970. The foundations were laid by Freudenthal and his colleagues at the former IOWO, the oldest predecessor of the Freudenthal Institute. The actual impulse for the reform movement was the inception, in 1968, of the Wiskobas project, initiated by Wijdeveld and Goffree. The project's first merit was that Dutch mathematics education was not affected by the New Math movement. The present form of RME has been mostly determined by Freudenthal's (1977) view on mathematics. He felt mathematics must be connected to reality, stay close to children's experience and be relevant to society, in order to be of human value. Instead of seeing mathematics as a subject to be transmitted, Freudenthal stressed the idea of mathematics as a human activity. Mathematics lessons should give students the 'guided' opportunity to 're-invent' mathematics by doing it. This means that in mathematics education, the focal point should not be on mathematics as a closed system but on the activity, on the process of mathematization (Freudenthal, 1968).

[6] One difference in emphasis is, for instance, that some representatives of RME put more emphasis on constructive learning and some put more on the point of view of re-constructive teaching.

[7] For more about development and research related to RME, see the paper by Koeno Gravemeijer presented at this conference.

Later on, Treffers (1978, 1987) explicitly formulated the idea of two types of mathematization in an educational context; he distinguished ‘horizontal’ and ‘vertical’ mathematization. In broad terms, these two types can be understood as follows. In horizontal mathematization, the students come up with mathematical tools which can help to organize and solve a problem set in a real-life situation. Vertical mathematization is the process of reorganization within the mathematical system itself, for instance, finding shortcuts and discovering connections between concepts and strategies and then applying these discoveries. Thus horizontal mathematization involves going from the world of life into the world of symbols, while vertical mathematization means moving within the world of symbols (see also Freudenthal, 1991). Although this distinction seems to be free from ambiguity, Freudenthal stated that it does not mean that the difference between these two worlds is clear cut. He also stressed that these two forms of mathematization are of equal value. Furthermore, one must keep in mind that mathematization can occur at different levels of understanding.

Misunderstanding of ‘realistic’

Despite this clear statement about horizontal and vertical mathematization, RME became known as ‘real-world mathematics education.’ This was especially true outside the Netherlands, but the same interpretation can also be found within the Netherlands. It must be acknowledged that the name ‘Realistic Mathematics Education’ is somewhat confusing in this respect.

The reason, however, why the Dutch reform of mathematics education was called ‘realistic’ is not just because of its connection with the real world, but is related to the emphasis that RME puts on offering the students problem situations which they can imagine. The Dutch translation of ‘to imagine’ is ‘zich REALISEren.’ It is this emphasis on making something real in your mind, that gave RME its name. For the problems presented to the students, this means that the context can be one from the real world but this is not always necessary. The fantasy world of fairy tales and even the formal world of mathematics can provide suitable contexts for a problem, as long as they are real in the student’s mind.

3 How the Dutch try to teach arithmetic in primary school

RME reflects a certain view on mathematics as a subject, on how children learn mathematics and on how mathematics should be taught (Van den Heuvel-Panhuizen, 1996). These views can be characterized by the following six principles.[8] Some of them originate more from the point

[8] This list of principles is an adapted version of the five tenets of the framework for the RME instruction theory distinguished by Treffers (1987): ‘phenomenological exploration by means of contexts’, ‘bridging by vertical instruments’, ‘pupils’ own constructions and productions’, ‘interactive instruction’, and ‘intertwining of learning strands.’ The first three principles described in this section have significant consequences for RME assessment (see Van den Heuvel-Panhuizen, 1996).

of view of learning and some are more closely connected to the teaching perspective. The list below is a mix of principles each reflecting a part of the identity of RME.

1. *Activity principle*

The idea of mathematization clearly refers to the concept of mathematics as an activity which, according to Freudenthal (1971, 1973), can best be learned by doing (see also Treffers, 1978, 1987). The students, instead of being receivers of ready-made mathematics, are treated as active participants in the educational process, in which they develop all sorts of mathematical tools and insights by themselves. According to Freudenthal (1973), using scientifically structured curricula, in which students are confronted with ready-made mathematics, is an 'anti-didactic inversion.' It is based on the false assumption that the results of mathematical thinking, placed in a subject-matter framework, can be transferred directly to the students.
The activity principle means that students are confronted with problem situations in which, for instance, they can produce fractions and gradually develop an algorithmic way of multiplication and division, based on an informal way of working. In relation to this principle, 'own productions' play an important role in RME.

2. *Reality principle*

As in most approaches to mathematics education, RME aims at enabling students to apply mathematics. The overall goal of mathematics education is that students must be able to use their mathematical understanding and tools to solve problems. This implies that they must learn 'mathematics so as to be useful' (see Freudenthal, 1968).
In RME, however, this reality principle is not only recognizable at the end of the learning process in the area of application, reality is also conceived as a source for learning mathematics. Just as mathematics arose from the mathematization of reality, so must learning mathematics also originate in mathematizing reality. Even in the early years of RME it was emphasized that if children learn mathematics in an isolated fashion, divorced from their experiences, it will quickly be forgotten and the children will not be able to apply it (Freudenthal, 1971, 1973, 1968). Rather than beginning with certain abstractions or definitions to be applied later, one must start with rich contexts demanding mathematical organization or, in other words, contexts that can be mathematized (Freudenthal, 1979, 1968). Thus, while working on context problems, the students can develop mathematical tools and understanding.

3. *Level principle*

Learning mathematics means that students pass through various levels of understanding: from the ability to invent informal context-related solutions, to the creation of various levels of short cuts and schematizations, to the acquisition of insight into the underlying principles and the discernment of even broader relationships. The condition for arriving at the next level is the ability to reflect on the activities conducted. This reflection can be elicited by interaction. Models serve as an important device for bridging this gap between informal, context-related mathematics and more formal mathematics. First, the students develop strategies closely connected to the context. Later on, certain aspects of the context situation can become more general, which means that the context more or less acquires the character of a model and as such can give support for solving other, but related, problems. Eventually, the models give the students access to more formal mathematical knowledge. In order to fulfil the bridging function between the informal and formal levels, models have to shift from a 'model of' a particular situation to a 'model for' all kinds of other, but equivalent, situations.[9]

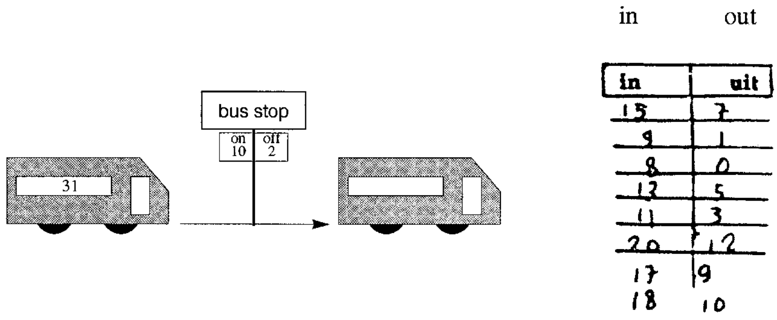

Figure 2: At the bus stop (from Streefland, 1996, p. 15 and 16)

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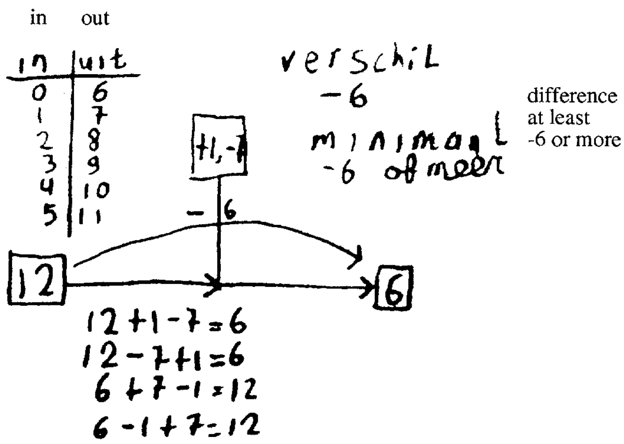

Figure 3: Two number sentences (from Streefland, 1996, p. 17)

The bus context[10] is an example from 'daily life' that can evolve to a more general and formal level. In the beginning, an illustration is used to describe the changes at the bus stop (see Figure 2). Later on the bus context becomes a 'model for' understanding all kinds of number

[9] It was Streefland who, in 1985, detected the shift in models as a crucial mechanism in the growth of understanding. Later on, this idea of a shift from the 'model of' to the 'model for' became a significant element within RME thinking about progress in students' understanding of mathematics (see Streefland, 1991; Treffers, 1991; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1995).

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sentences, and then the students can go far beyond the real bus context. They can even use the model for backwards reasoning (see the last two sentences in Figure 3).
An important requirement for having models functioning in this way is that they are rooted in concrete situations and that they are also flexible enough to be useful in higher levels of mathematical activities. This means that the models will provide the students with a foothold during the process of vertical mathematization, without obstructing the path back to the source.

The strength of the level principle is that it guides growth in mathematical understanding and that it gives the curriculum a longitudinal coherency. This long-term perspective is characteristic of RME. There is a strong focus on the relation between what has been learned earlier and what will be learned later. A powerful example of such a 'longitudinal' model is the number line. It begins in first grade as (a) a beaded necklace on which the students can practice all kind of counting activities. In higher grades, this chain of beads successively becomes (b) an empty number line for supporting additions and subtractions[11], (c) a double number line for supporting problems on ratios, and (d) a fraction/percentage bar for supporting working with fractions and percentages (see Figure 4).

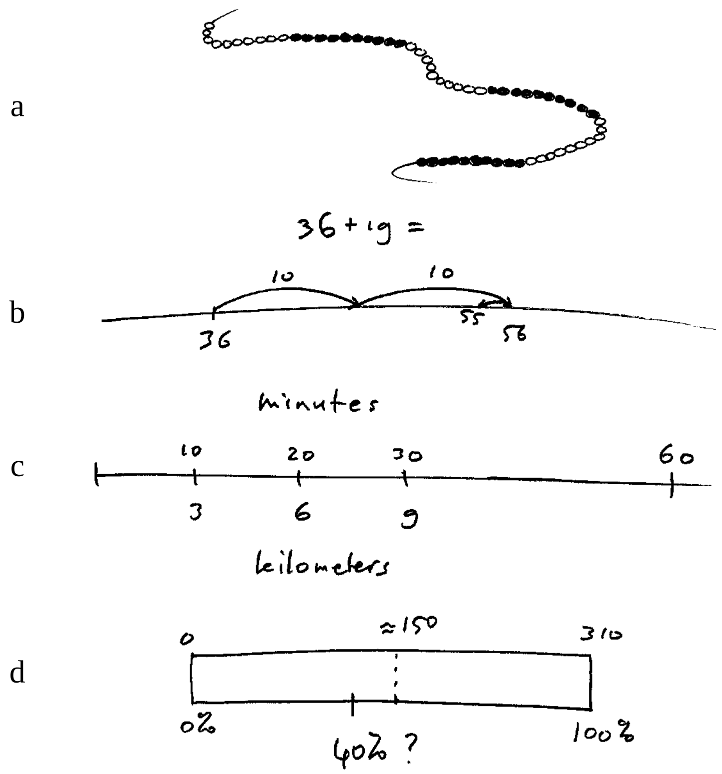

Figure 4: Different ways the number line can appear

4. Inter-twinement principle

It is also characteristic of RME that mathematics, as a school subject, is not split into distinctive

[11] See the paper presented at this conference by Julie Menne about her 'Jumping ahead' program for under-achievers in the early grades.

learning strands. From a deeper mathematical perspective, the chapters within mathematics cannot be separated. Moreover, solving rich context problems often means that you have to apply a broad range of mathematical tools and understandings. For instance, if children have to estimate the size of the flag, pictured in Figure 5, this estimation involves not only measurements but also ratio and geometry.

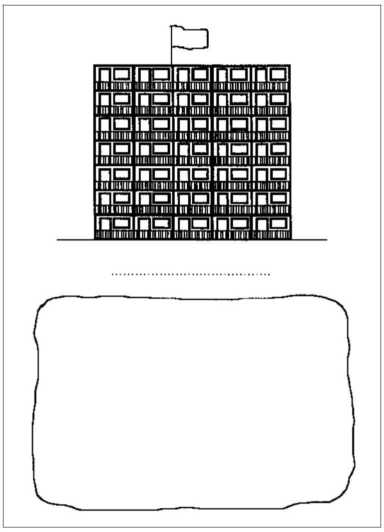

Figure 5: Flag problem

In the same way, the mirror activity in Figure 6 clearly shows how geometry and early arithmetic can go together.

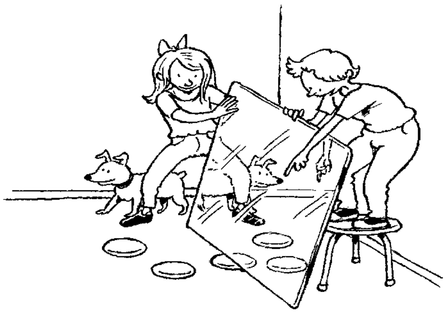

Figure 6: Mirroring and counting
(from Treffers, Van den Heuvel-Panhuizen, and Buys (Eds.), 1999)

The strength of the inter-twinement principle is that it renders coherency to the curriculum. This principle involves not only the mutual relationship between the different chapters of mathematics but can also be found in the different parts of one chapter. In the number strand, for instance, topics like number sense, mental arithmetic, estimation and algorithms are closely related; this issue is considered in more detail in a later section.

5. Interaction principle

Within RME, the learning of mathematics is considered as a social activity. Education should offer students opportunities to share their strategies and inventions with each other. By listening to what others find out and discussing these findings, the students can get ideas for improving their strategies. Moreover, the interaction can evoke reflection, which enables the students to reach a higher level of understanding.

The significance of the interaction principle implies that whole-class teaching plays an important role in the RME approach to mathematics education. However, this does not mean that the whole class is proceeding collectively and that every student is following the same track and is reaching the same level of development at the same moment. On the contrary, within RME, children are considered as individuals, each following an individual learning path. This view on learning often results in pleads for splitting up classes into small groups of students each following their own learning trajectories. In RME, however, there is a strong preference for keeping the class together as a unit of organization[12] and for adapting the education to the different ability levels of the students instead. This can be done by means of providing the students with problems which can be solved on different levels of understanding.

6. Guidance principle

One of Freudenthal's (1991) key principles for mathematics education is that it should give students a 'guided' opportunity to 're-invent' mathematics. This implies that, in RME, both the teachers and the educational programs have a crucial role in how students acquire knowledge. They steer the learning process, but not in a fixed way by demonstrating what the students have to learn. This would be in conflict with the activity principle and would lead to pseudo-understanding. Instead, the students need room to construct mathematical insights and tools by themselves. In order to reach this desired state, the teachers have to provide the students with a learning environment in which the constructing process can emerge. One requirement is that teachers must be able to foresee where and how they can anticipate the students' understandings and skills that are just coming into view in the distance (see also Streefland, 1985). Educational programs should contain scenarios which have the potential to work as a lever in shifting students' understanding. It is important for these scenarios that they always hold the perspective of the long-term teaching/learning trajectory based on the desired goals. Without this perspective, it is not possible to guide the students' learning.

Whereas on the micro-didactic level RME has a lot in common with the constructivistic approach to mathematics education, on the macro-didactic level of the curriculum some major differences between the two become apparent. As a matter of fact, the constructivistic approach does not have a macro-didactic level in which decisions are made about the goals for education and the teaching/learning trajectories which need to be covered in order to reach these goals. In contrast with RME, the constructivistic approach is more a learning theory than a theory of education. The guidance principle leads to the curriculum ideas of RME.

[12] Within the structure of keeping the group together, a variety of teaching methods can be applied: ranging from whole-class teaching, to group work to individual work.

4 What are the determinants of the Dutch mathematics curriculum?

Unlike many other countries, at primary school level the Netherlands does not have a centralized decision making regarding curriculum syllabi, text books or examinations (see Mullis et al., 1997). None of these need approval by the Dutch government. For instance, the schools can decide which textbook series they use. They can even develop their own curriculum. In general, what is taught in primary schools is, for the greater part, the responsibility of teachers and school teams and the teachers are fairly free in their teaching. To give some more examples, teachers have a key to the school building, they are allowed to make changes in their timetable without asking the director of the school (who often teaches a class too), and, as a last example, the teacher’s advice at the end of primary school, rather than a test, is the most important criterion for allocating a student to a particular level of secondary education.

Despite this freedom in educational decision making — or probably one should say thanks to the absence of centralized educational decision making — the mathematical topics taught in primary schools do not differ a lot between schools. In general, all the schools follow the same curriculum. This leads to the question: what determines this curriculum?

Until recently, there were three important determinants for macro-didactic tracking in Dutch mathematics education in primary school:

- the mathematics textbooks series
- the ‘Proeve’; a document recommending the mathematical content top be taught in primary school
- the key goals to be reached by the end of primary school as described by the government.

4.1 The determining role of textbooks

In the current world-wide reform of mathematics education, speaking about textbooks — not to mention the use of them — often elicits a negative association. In fact, many reform movements are aimed at getting rid of textbooks. In the Netherlands, however, the contrary is the case. Here, the improvement of mathematics education depends largely on new textbooks. They play a determining role in mathematics education. Actually, textbooks are the most important tools guiding the teachers’ teaching. This is true of both the content and the teaching methods, although with regard to the latter, the guidance provided is not sufficient to reach all teachers. Many studies revealed indications that the implementation of RME in classroom practice is still not optimal Gravemeijer et al., 1993; Van den Heuvel-Panhuizen and Vermeer, 1999).

The determining role of textbooks, however, does not mean that Dutch teachers are prisoners of their textbooks. As already stated, Dutch teachers are fairly free in their teaching and the schools can decide which textbook series they use. Currently, about eighty percent of Dutch primary schools use a mathematics textbook series inspired by RME to a greater or lesser degree. Compared to ten or fifteen years ago this percentage has changed significantly. At that time, only half the schools worked with such a textbook series (De Jong, 1986). A textbook series is developed by commercial publishers.[13] The textbook authors are independent developers of mathematics education, but they can use the ideas for teaching activities resulting from the developmental research at the Freudenthal Institute (and its predecessors) and at the SLO, the Dutch Institute for Curriculum Development.

4.2 The ‘Proeve’ — a domain description of primary school mathematics

An important aid in the development of textbooks is also the guidance which, since the mid-eighties, has come from a series of publications, called the ‘Proeve.’[14] Treffers is the main author of this series. The publications contain descriptions of the various domains within mathematics as a school subject. Work on the ‘Proeve’ is still going on and eventually there will be descriptions for all the basic number skills, written algorithms, ratios and percentages, fractions and decimal numbers, measurement, and geometry. Although the ‘Proeve’ is written in an easy style with many examples, it is not written as a series for teachers. Instead, it is intended as support for textbook authors, teacher educators and school advisors. On the other hand, many such experts on mathematics education were, and still are, significant contributors to the realization of this series.

Looking back at the Dutch reform movement in mathematics education, it can be concluded that the reform proceeded in an interactive and informal way, without government interference. Instead developers and researchers, in collaboration with teacher educators, school advisors and teachers, worked out teaching activities and learning strands. Later on, these were included in textbooks.

4.3 The key goals for mathematics education

Until recently there was no real interference from the Dutch government regarding the content of educational programs. There was only a general law containing a list of subjects to be taught.

[13] More about one particular textbook series will be presented by Kees Buys in his workshop at this conference.

[14] The complete title of this series is ‘Design of a national program for mathematics education in primary schools’ [Proeve van een Nationaal Programma voor het reken- wiskundeonderwijs op de basisschool]. The first part of this series was published in 1989 (see Treffers, De Moor and Feijs, 1989). Note that the title refers to a ‘national program’ although there was no government interference. The authors liked to label this a national program in order to achieve a common program; in this aim they have succeeded.

What topics had to be taught within these subjects was almost entirely the responsibility of the teachers and the school teams. A few years ago, however, government policy changed and, in 1993, the Dutch Ministry of Education came up with a list of attainment targets, called ‘Key Goals.’ These goals describe what students have to learn in each subject by the end of their primary school career (at age twelve years). For mathematics the list contains 23 goals, split into six domains (see Table 1). The content of the list is in agreement with the ‘Proeve’ documents previously mentioned.

Compared to goal descriptions and programs from other countries it is notable that some widespread mathematical topics are not mentioned in this list, for instance, problem solving, probability, combinatorics, and logic. Another striking feature of the list is that it is so simple. This means that the teachers have a lot of freedom in interpreting the goals. At the same time, however, such a list does not give much support to teachers. As a result, the list is actually a ‘dead’ document, mostly put away in a drawer when it arrives at school. Nevertheless, this first list of key goals was important for Dutch mathematics education. The government’s publication of the list confirmed and, in a way, validated the recent changes in the Dutch curriculum.

The predominant changes were:

- more attention was to be paid to mental arithmetic and estimation
- formal operations with fractions were no longer in the core curriculum, the students now only have to do operations with fractions in context situations
- geometry was officially included in the curriculum
- and also the insightful use of a calculator.

However, not all these changes have been included in the textbooks and implemented in present classroom practice. This is especially true for geometry and the use of a calculator.

In the years since 1993, there have been discussions about these 23 key goals (see De Wit, 1997). Almost everybody is agreed that these goals can never be sufficient to support improvements in classroom practice nor to control the outcome of education. The latter is conceived by the government as a powerful tool for safeguarding the quality of education. For both purposes, the key goals were judged as failing. Simply stating goals is not enough in order to achieve them. For testing the outcome of education the key goals are also inappropriate. Many complaints were heard that the goals were not formulated precisely enough to provide yardsticks for testing. These arguments were heard not only for mathematics, but also for all the primary school subjects for which key goals had been formulated.

Table 1: Key goals for Dutch primary school students in mathematics

		By the end of primary school, the students:
General abilities	1	Can count forward and backward with changing units
	2	Can do addition tables and multiplication tables up to ten
	3	Can do easy mental-arithmetic problems in a quick way with insight in the operations
	4	Can estimate by determining the answer globally, also with fractions and decimals
	5	Have insight into the structure of whole numbers and the place-value system of decimals
	6	Can use the calculator with insight
	7	Can convert simple problems which are not presented in a mathematical way into a mathematical problem
Written algorithms	8	Can apply the standard algorithms, or variations of these, to the basic operations, of addition, subtraction, multiplication and division in simple context situations
Ratio and percentage	9	Can compare ratios and percentages
	10	Can do simple problems on ratio
	11	Have an understanding of the concept percentage and can carry out practical calculations with percentages presented in simple context situations
	12	Understand the relation between ratios, fractions, and decimals
Fractions	13	Know that fractions and decimals can stand for several meanings
	14	Can locate fractions and decimals on a number line and can convert fractions into decimals; also with the help of a calculator
	15	Can compare, add, subtract, divide, and multiply simple fractions in simple context situations by means of models
Measurement	16	Can read the time and calculate time intervals; also with the help of a calendar
	17	Can do calculations with money in daily-life context situations
	18	Have an insight into the relation between the most important quantities and corresponding units of measurement
	19	Know the current units of measurement for length, area, volume, time, speed, weight, and temperature, and can apply these in simple context situations
	20	Can read simple tables and diagrams, and produce them based on own investigations of simple context situations
Geometry	21	Have mastered some basic concepts with which they can organize and describe a space in a geometrical way
	22	Can reason geometrically using building blocks, ground plans, maps, pictures, and data about positioning, direction, distance, and scale
	23	Can explain shadow images, can compound shapes, and can devise and identify cut-outs of regular objects

4.4 Blueprints of longitudinal teaching/learning trajectories — a new factor for macro-didactic tracking

For several years it was unclear which direction would be chosen for improving the key goals: either providing a more detailed list of goals for each grade expressed in operationalized terms, or a description which supports teaching rather than pure testing. In 1997, the government tentatively opted for the latter and asked the Freudenthal Institute to work it out for mathematics. This decision resulted in the start of the TAL Project[15] in September 1997. The purpose of this project, which the Freudenthal Institute is carrying out together with the SLO and CED[16], is to contribute to the enhancement of classroom practice, starting with that in the early grades. The reason for starting in the lower grades was that, at the same time, the government took measures to reduce the class size in these grades. The products of the TAL Project might eventually become the fourth guiding factor for the macro-didactic tracking in Dutch primary school mathematics education.

A trajectory blueprint on whole numbers as a start

The first focus of the project was on the development of a description of a longitudinal teaching/learning trajectory on whole-number arithmetic. The first description for the lower grades (including K1, K2, and grades 1 and 2)[17] was published in November 1998. The definitive version followed a year later (Treffers, Van den Heuvel-Panhuizen, and Buys (Eds.), 1999). Now the focus is on a continuation of a whole-number trajectory for the higher grades of primary school (including grades 3 through 6) and recently a start has been made on developing a teaching/learning trajectory for measurement and geometry. In the future, other strands will be worked out for fractions, decimals and percentages.

In the whole-number trajectory, arithmetic is interpreted in a broad sense, including number knowledge, number sense, mental arithmetic, estimation and algorithms. In fact the description is meant to give an overview of how all these number elements are related to each other, both in a longitudinal and in a cross-sectional way.

A new approach to goal description as a framework to support teaching

The stepping stones that students will pass (in one way or another) on their way to reaching the goals at the end of primary school are crucial in the trajectory blueprints. These stepping stones can be seen as intermediate goals. As goals, however, they differ in many respects from the usual end-of-grade goal descriptions which are mostly very rigid in order to be suitable as a direct basis for testing. In fact, in several respects the intended blueprints of the teaching/learning trajectories are the contrary of the goal descriptions which are traditionally supposed

[15] TAL is a Dutch abbreviation that stands for Intermediate Goals Annex Teaching/Learning Trajectories.
[16] SLO is the Dutch Institute for Curriculum Development. CED is the school advisory center for the city of Rotterdam. In the near future other institutes will probably also officially participate in the TAL Project.
[17] These grades cover students from 4 to 8 years old.

to guide education. Instead of unambiguous goal descriptions in behavioral terms, the teaching/ learning trajectory provides the teachers with a more or less narrative sketch of how the learning process can proceed provided that a particular educational setting is realized. The description contains many examples, including video-taped examples on a CD-Rom, of student behavior and student work in connection with core activities in teaching.

Giving the teachers a pointed overview of how children's mathematical understanding can develop from grades K1 through 2 (and eventually through 6) and of how education can contribute to this development is the main purpose of this alternative to the traditional focus on clear goals as the most powerful engine for enhancing classroom practice. In no way, however, are the trajectory blueprints meant as recipe books. They are rather intended to provide teachers with a mental educational map which can help them, if necessary, to make adjustments to the textbook. Another difference with the traditional goal description is that there is no strict pre-requisite structure. In addition, the learning processes are not regarded as a continuous process of small steps, nor are the intermediate goals considered as a check list to see how far students have got. As a matter of fact, such an approach neglects the discontinuities in the learning process and does not take into account the degree to which understanding and skill performance are determined by the context and how much they differ between individuals. Instead of a check list of isolated abilities, the trajectory blueprints try to make clear how the abilities are built up in connection with each other. What is learned in one stage is understood and performed on a higher level in a following stage.

The binding force of levels and their didactical use

It is this level characteristic of learning processes, which is a constitutive element of RME, that brings longitudinal coherency into the teaching/learning trajectory. Another crucial implication of this level characteristic is that students can understand something on different levels. In other words, they can work on the same problems without being on the same level of understanding. The distinction of levels in understanding, which can have different appearances for different sub-domains within the whole number strand, is very fruitful for working on the progress of children's understanding. It offers footholds for stimulating this progress.

Levels in counting as a first example

As an example, one might consider the levels in counting[18] that have been distinguished for the early stage of the development of number concept in kindergarten and the beginning of grade 1. The following three levels have been identified (see Treffers, Van den Heuvel-Panhuizen, and Buys (Eds.), 1999):

- context-related counting
- object-connected counting
- (towards) a more formal way of counting.

[18] To indicate that there is no strict division between counting and calculating, in the teaching/ learning trajectory, the skill of counting is called 'counting-and-calculating.'

To explain this level distinction and to give an idea of how it can be used for making problems accessible to children and for eliciting shifts in ability levels, one can think of the ability of resultative counting up to ten. What if a child does not make any sense of the 'how-many' question (see Figure 7)? Does this mean that the child is simply not able to do resultative counting?

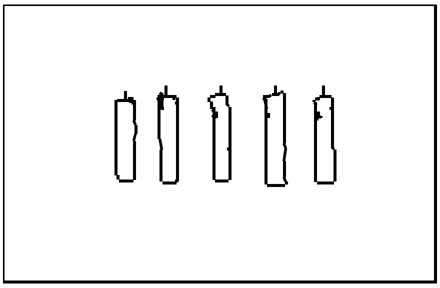

Figure 7: How many ...?

That this is not necessarily the case may become clear if the teacher moves to a context-related question. This means that a context-related question is asked instead of a plain 'how-many' question, e.g.

- how old is she (while referring to the candles on a birthday cake)? (see Figure 8) and
- how far can you jump (while referring to the dots on a dice)?
- how high is the tower (while referring to the blocks of which the tower is built)?

:

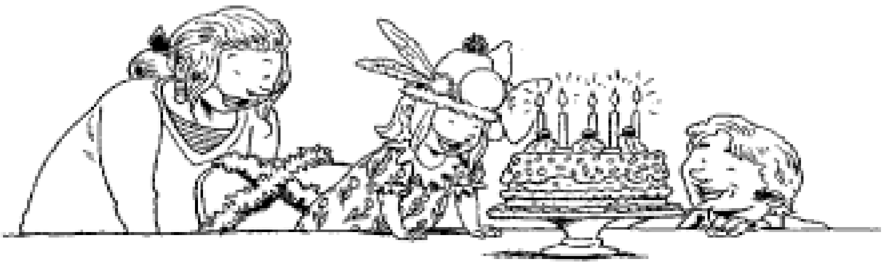

Figure 8: How old ...?
(from Treffers, Van den Heuvel-Panhuizen, and Buys (Eds.), 1999, p.26)

In the context-related questions, the context gives meaning to the concept of number. This context-related counting precedes the level of the object-related counting in which the children can handle the direct 'how many' question in relation to a collection of concrete objects without any reference to a meaningful context. Later on, the presence of the concrete objects is also no longer needed to answer 'how many' questions. Via symbolizing, the children have reached a level of understanding in which they are capable of what might be called formal counting, which means that they can reflect upon number relations and that they can make use of this knowledge.

Levels in calculating as a second example

With regard to the field of early calculating in grade 1 (with numbers up to 20) the following three levels have been identified (see Treffers, Van den Heuvel-Panhuizen, and Buys (Eds.), 1999):

- calculating by counting (calculating 7+6 by laying down seven on-guilder coins and six one-guilder coins and counting the total one by one)
- calculating by structuring (calculating 7+6 by laying down two five-guilder coins and three one-guilder coins)
- formal (and flexible) calculating (calculating 7+6 without using coins and by making use of ones knowledge about 6+6).

What these levels look like in classroom practice will be shown in the Restaurant lesson discussed in the next section (see §5.1).

In the higher grades when the students are doing calculations on a formal level, the levels mentioned already can be recognized in the three different calculation strategies for additions and subtraction up to 100:

- the jumping strategy (this strategy is related to calculating by counting; it implies keeping the first number as a whole number: 87-39= .... 87-30=57; 57-7=50; 50-2=48)
- the strategy of splitting numbers in tens and ones (this strategy is related to calculating by structuring; it implies making use of the decimal structure: 87-39= .... 80-30=50; 7-7=0; 50-2=48)
- flexible calculating (this strategy implies making use of knowledge of number relations and properties of operations: 87-39= .... 87-40=47; 47+1=48).

Didactical levels

Insight into these didactical levels provides teachers with a powerful mainstay for gaining access to children's understanding and for working on shifts in understanding. After starting, for instance, with context-related questions ('how old is she?') the teacher can gradually push back the context and reach the object-related questions ('how many candles are on the birthday cake?'). The level categories for calculations up to 20 and 100 differ remarkably from, for instance, levels based on problem types[19] and levels based on the size of the numbers to be processed. They also deviate from the more general concrete–abstract distinctions in levels of understanding[20] and from level distinctions ranging from material-based operating with numbers to mental procedures; with verbalizing as an intermediate state[21]. The ideas with which the TAL levels in the early grades identify most can be found in the work of Donaldson (1978) and Hughes (1986).[22]

[19] The CGI problem types might be considered as an example of a level distinction based on types of problems.
[20] Like the Piagetian levels in cognitive growth.
[21] Like the levels distinguished by Gal'perin.

So far we have discussed some of the main ideas behind the trajectory blueprints. As already stated, the TAL Project is only just starting work on them. It is not yet known how they will function in school practice and whether they can really help teachers. Inquiries made so far (De Goeij, Nelissen, and Van den Heuvel-Panhuizen, 1998; Groot, 1999; Slavenburg and Krooneman, 1999), however, give us a general feeling that they may indeed help teachers and that the TAL teaching/learning trajectory on whole number arithmetic for the lower primary grades has triggered something that in one way or another can bring not only the children but also Dutch mathematics education to a higher level.

The TAL team were interested to discover that making a trajectory blueprint was not only a matter of writing down what was already known in a popular and accessible way for teachers, but that the work on the trajectory also resulted in new ideas emerging about teaching mathematics and revisiting our current thinking about teaching.

5 Three examples of RME classroom practice

The answer to the 'what' question about the Dutch arithmetic curriculum will be restricted to giving an impression of it based on the following three problems:

- $6 + 7 =$
- $81 \div 6 =$
- $4 \times f\ 1.98 =$

It will be clear that this selection in no way covers the complete Dutch arithmetic curriculum. It is just a brief overview in giant steps: one problem will be shown for the early grades, one for the middle grades, and one for the end grades of primary school. What these examples have in common is that they can all be solved on different levels and can thus be fertile soil for progress. They also make clear that the 'what' question about the Dutch arithmetic curriculum can never be disconnected from the 'how.'

5.1 Restaurant

The Restaurant lesson is one of the lessons on the CD-Rom and belongs to the TAL teaching/learning trajectory on whole numbers for the lower grades. The lesson was videotaped in a mixed class containing K2 and grade 1 children, aged five and six years. The teacher, Ans Veltman, is one of the staff members of the TAL team. She also designed the lesson, although

[22] Nevertheless there are also some significant differences between their ideas and the TAL ideas about levels. Donaldson, for instance, did not apply her ideas to counting and Hughes did not identify what is called context-related counting in TAL.

she would disagree with this — Ans feels that her student Maureen was the developer of this lesson. Maureen opened a restaurant in a corner of the classroom and everybody was invited to have a meal. The menu card shows the children what they can order and what it costs. The prices are in whole guilders (see Figure 9).

Figure 9: The menu card for Maureen's restaurant

The teacher's purpose with this lesson is to work on a difficult addition problem bridging the number ten. The way she does this, however, reflects a world of freedom for the students. The teacher announced that two items could be chosen from the menu and asked the children what items they would chose and how much this would cost. In other words, it appeared as if there was no guidance from the teacher, but the contrary was true. By choosing a pancake and an ice cream, costing 7 guilders and 6 guilders, respectively, she knew in advance what problem the class would be working on; namely the problem of adding above ten, which she is what she wanted them to work on.

There is a purse with some money to pay for what is ordered and the teacher had arranged that the purse should contain five-guilder coins and one-guilder coins. (This again shows subtle guidance from the teacher.) Then the students start ordering. Niels chooses a pancake and an ice cream. Jules writes it up on a small blackboard. The other children shout: 'Yeah ... me too.' They agree with Niels' choice. Then the teacher asks what this choice would cost in total.

Here is a summary of what the children did:

- Maureen counted 13 one-guilder coins. Six coins for the ice cream and seven coins for the pancake (calculating by counting) (see Figure 10).

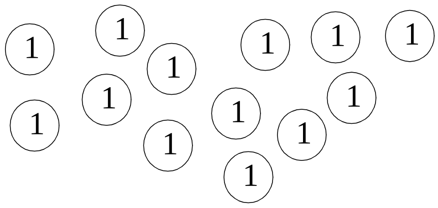

Figure 10: Maureen's strategy

- Thijs and Nick changed five one-guilder coins for one five-guilder coin and pay the ice cream with "5" and "1" and the pancake with "5" and "1" and "1". Then they saw that the two fives make ten and the three ones make 13 in total (calculating by structuring) (see Figure 11).

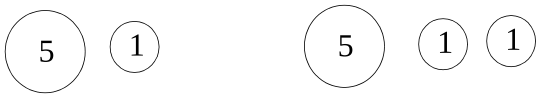

Figure 11: Thijs' strategy

Later Nick placed the coins in a row: "5", "5", "1", "1", "1".

- Luuk came up with the following strategy: "First put three guilders out of the six to the seven guilders, that makes ten guilders, and three makes thirteen" (calculating by structuring and towards formal calculating).

- Hannah did not make use of the coins. She calculated: "6 and 6 makes 12, and 1 makes 13 guilders. Another student came with: "7 and 7 makes 14, minus 1 makes 13" (formal and flexible calculating).

This Restaurant lesson makes clear that children who differ in skill and level of understanding can work in class on the same problem. To do this, it is necessary that problems that can be solved on different levels are presented to the children. The advantage for the students is that sharing and discussing their strategies with each other can function as a lever to raise their understanding. The advantage for teachers is that such problems can provide them with a cross-section of their class's understanding at any particular moment, and also gives them a longitudinal overview of the trajectory they need to go along. The cross-section of strategies at any moment indicates what is coming within reach in the immediate future. As such, this cross-section of strategies contains handles for the teacher for further instruction.

5.2 Parents' evening

The next classroom vignette[23] shows how the RME principles can contribute to growth in mathematical understanding. The starting point of the lesson is the exploration of a context problem which, and this is essential, can be solved on several levels of understanding. By discussing and sharing solution strategies in class, the students who first solved the problem by means of a long-winded strategy can progress to a higher level of understanding. As a result of this process, which is called 'progressive mathematization', new mathematical concepts can be constituted.

The scene is set in a third-grade classroom. The students are eight through nine years old. The teacher starts with the presentation of a problem about a parents' evening that is being organized. The question is about the number of tables needed to seat the parents (see Figure 12).

'... Tonight, there will be a parents' evening ...
... The slips I received from you tell me that 81 persons will attend ...
... The meeting will take place in the large hall ...
... The parents will be seated at large tables ...
... Six persons can sit at each table...'

Then the teacher makes a drawing of such a table on the black board:

After doing this, the teacher asks:
'How many tables do we need for 81 persons?"

Figure 12: *Tables* problem

The students begin to work and the teacher walks around the classroom. Whenever necessary she gives some help. About ten minutes later she asks the students to show their work and explain their solutions.

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[23] This classroom activity originates from Van Galen et al. (1991) (see also Van Galen and Feijs, 1991); the present vignette was also used by De Lange in his plenary lecture at ICME 1996 in Seville, Spain.

Badr drew as many tables as he needed to have all the parents seated (see Figure 13).

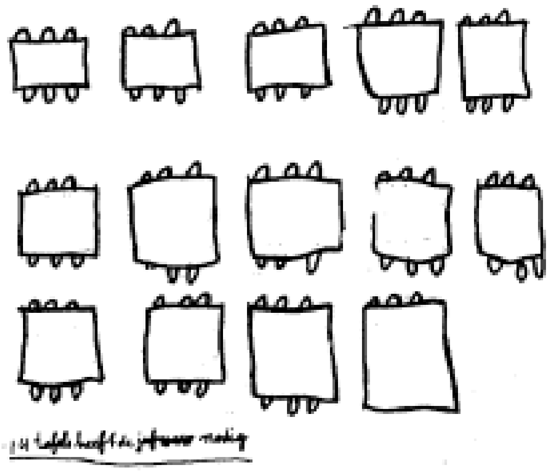


Figure 13: Badr’s work

Roy started in the same way, but after he drew two complete tables he drew two rectangles and put the number six in them. While he was drawing more of these rectangles he suddenly realized that if you have five tables, 30 parents can be seated. He continued drawing rectangles and after another five he wrote down 60. Then he drew another two, wrote down 72, and another one, and wrote down 78. He finished with a rectangle with the number 3 in it (see Figure 14).

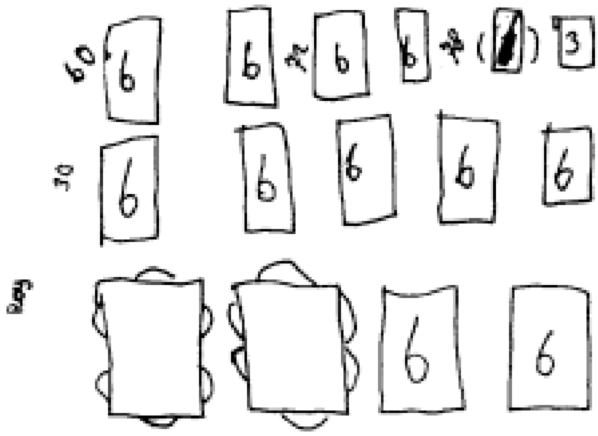
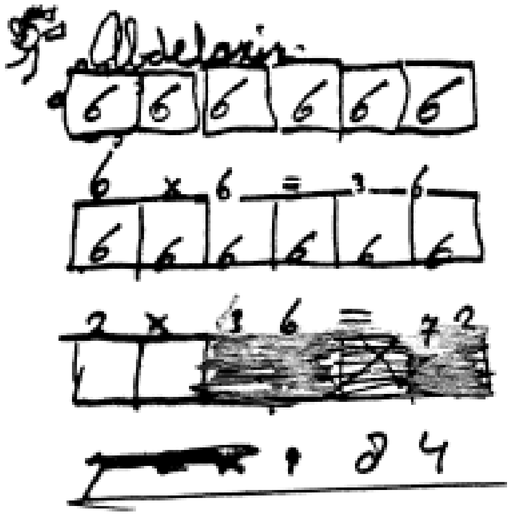


Figure 14: Roy’s work

A third student, Abdelaziz, was even more advanced in mathematizing the problem. Although he also started by drawing a copy of the table that was on the blackboard, he immediately moved up to a more formal solution by using his knowledge about multiples of six. He wrote down $6 \times 6 = 36$, doubled this number and came to 72, and then he added two more tables to the 72 and came to the answer of 84 (see Figure 15).
When you look at these three solutions, it is evident that some mathematization took place at each level, even in Badr’s work, since visualization and schematization are also powerful tools



Figuur 15: Abdelaziz’ work

for mathematizing. In the other two examples the mathematics is more visible, but not yet at the level that is aimed at. This problem was meant as a start for learning long division. In order to achieve this goal, the problem should be followed by other problems. Thus, after the class discussion about the different strategies had finished, the teacher presented a new problem: the *Coffee pots* problem (see Figure 16).

> *‘... The 81 parents will be offered a cup of coffee ...*
> *... ... You can fill 7 cups from one coffee pot...*
> *... How many coffee pots will be needed?’*

Figure 16: *Coffee pots* problem

From a mathematical point of view this problem is the same as the previous one. Instead of dividing by six, now, the students have to divide by seven. For these students, however, this is a completely different problem and it is also more difficult to make a visual presentation of it.

Tables are easier to draw than coffee pots although Badr tried to draw them (see Figure 17).

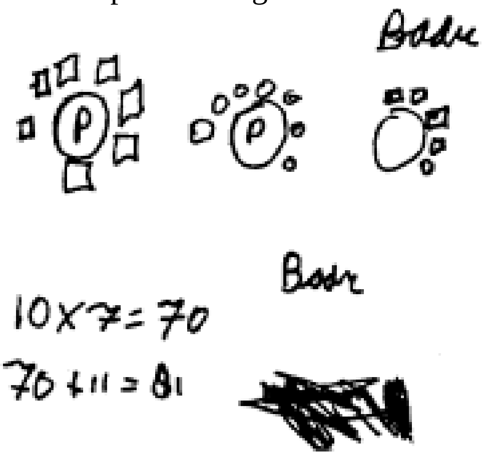


Figure 17: Badr's work on the *Coffee pots* problem

After he drew two pots he remembered the discussion about how the answer can be found more quickly by multiplying. He continued using 10 × 7 = 70 followed by 70 + 11 = 81, and decided that 12 pots are needed.

Both the constitution of mathematical tools (the representation of the problem situation, the schematization, the repeated addition, the application of number fact knowledge, the way of keeping track of the results, and the communicating about the strategies) and Badr's level shift were evoked by the problems given to students. More precisely, they were evoked by the cluster of problems. In a way, the context in this cluster of problems prompts them to re-invent the mathematics and to come to a higher level of understanding (see also Figure 20).

5.3 Buying loaves of bread

The ultimate overall goal of RME arithmetic is numeracy. Children should be able to make sense of numbers and numerical operations. Among other things, this implies that the children should be able to decide for themselves what calculation procedure is appropriate for solving a particular arithmetic problem. They should know when a mental calculation is adequate, when to use an estimate, and when it is better to do column arithmetic on paper or to use a calculator.[24] Being able to make decisions like these is one of the higher-order goals of arithmetic education to be reached in the higher grades of primary school. The *Buying loaves of bread* problem (see Figure 18) is very suitable for working on this goal.

[24] For more about mental and written strategies, see the paper presented at this conference by Meindert Beishuizen.

Lieke is buying four loaves of bread;
each costing ƒ 1.98.
Will one ten-guilder note be enough?

Figure 18: Buying loaves of bread problem

The student work in Figure 19 shows a wide variety of categories of calculation procedures which can be applied to solve this problem. Students A, B and C solved it in one way or another by column arithmetic. Students D, E and F[25] used an estimation method. Within these two major categories, the students applied several different strategies.

[25] This is the solution Adri Treffers added to the collection in his own handwriting.

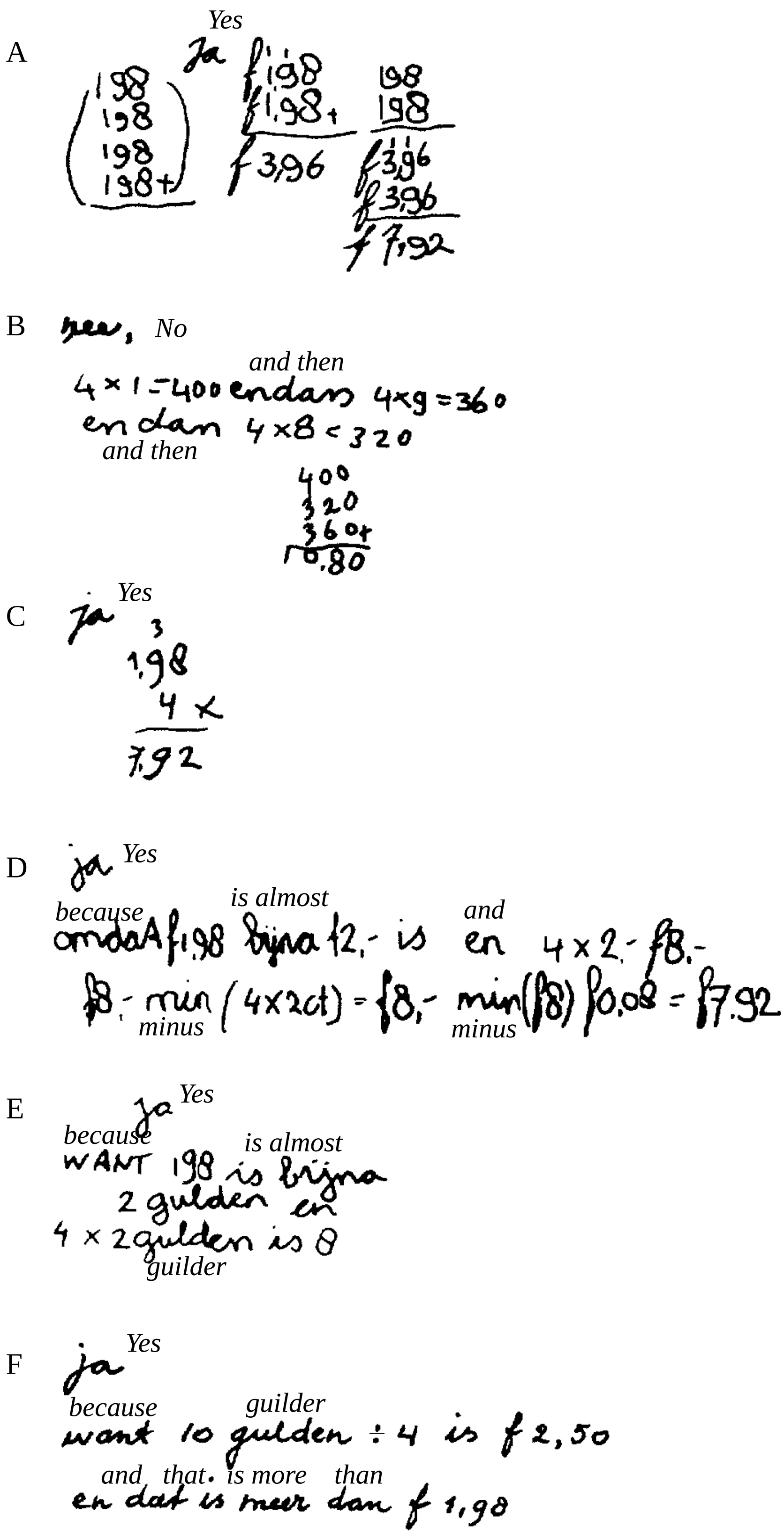


Figure 19: Student work *Buying loaves of bread* problem (translation added by author)

The need for class discussion
As in the *Parents' evening* and the *Restaurant problems*, a class discussion about how the students tackled the *Buying loaves of bread* problem can help them to make a more suitable choice next time. Especially if estimates are the issue, such a discussion can help to free the students from their idea that their working with numbers can only be called 'mathematics' if they make precise calculations. In problems like the *Buying loaves of bread* no precise calculation is needed, an estimate will do. And on the other hand, making an estimate to solve this problem can open the road to smart precise calculation! Student D's work is a good example of this.

Clusters of problems
In order to open the road to smart precise calculation, the global question of 'Enough money?' is followed by a more precise question like 'How much is left or how much more do you need?' As in the *Parents' evening* problem, this sequence of questions can be seen as a mini-trajectory with the potential to work as a lever to achieve shifts in the students' understanding. In a way, these clusters of problems are educational scenarios which guide the teaching and prompt the learning.

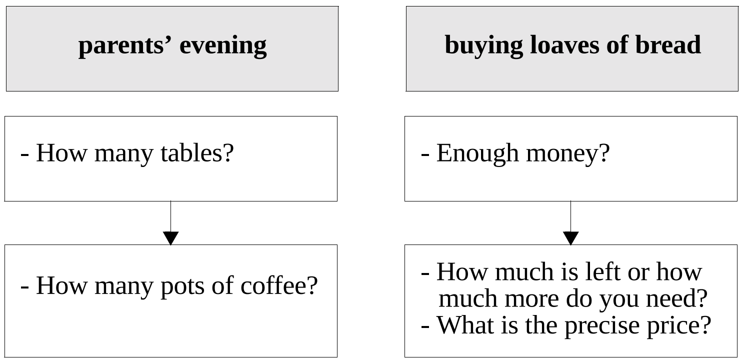


Figure 20: Clusters of problems as a didactical tool to evoke shifts in understanding

Coherence between calculation procedures
Apart from opening the students' eyes to making a suitable choice for a particular calculation procedure, the variety of work done on the *Buying loaves of bread* problem (see Figure 19) also provides the teacher with a powerful tool for discussing how the procedures are related to each other with the students. The work of students C and A shows perfectly that, instead of a column multiplication, a repeated column addition can be applied. Actually the latter is a preliminary stage of the standard written algorithm for multiplication. Student B's work even shows what comes before this preliminary stage: a kind of combination of horizontal arithmetic with whole-number value (instead of processing the numbers as digits) and column arithmetic.

This mixed procedure is a crucial element is the teaching/learning trajectory that the TAL team have in mind for arithmetic in the higher grades of primary school. Mental arithmetic is seen as the main branch, from which the column arithmetic branch is derived later. ‘Later’ means that, for instance, the standard algorithms for addition and subtraction (see Figure 21, +d and –d) are not on the agenda before grade 4. The algorithms for multiplication and division are not dealt with before grade 5. In the years before these grades, the students apply more or less shortened and mixed procedures based on a ‘horizontal’, whole-value way of calculating. Figure 21 shows how this way of calculating related to mental arithmetic gradually develops towards the standard algorithms. This is true at least for addition and multiplication. With regard to subtraction, a different track is followed. The subtraction algorithm does not arise naturally from the whole-value approach. Therefore, the addition algorithm might give access to the subtraction algorithm (see arrow in Figure 21). The most shortened standard algorithm for division is left out of the scheme. This algorithm no longer belongs to the core curriculum.

	a	b	c	d
+	257 585+ 700+130+12=842	257 585+ 700 130 12 842	257 585+ 12 130 700 842	1 1 257 585+ 842
–	937 685– 300-50+2=252	937 685– 300 –50 +2 252	-5 937 685– 302 50- 252	813 937 685– 252
x	87 9 x 63 720 783	87 9 x 783	87 60 x 5220	87 69 x 783 5220 6003
÷	32 / 7352 \\ 3200 100x 4152 3200 100x 952 320 10x 632 320 10x 312 160 5x 152 128 4x 24 229	32 / 7352 \\ 6400 200x 952 640 20x 312 288 9x 24 229	32 / 7352 \\ 200 6400 20 952 9 640 312 288 24 229	

Figure 21: Proposed[26] teaching/learning trajectory for whole number arithmetic in the higher primary grades: from horizontal to vertical calculation

[26] This scheme was developed in the early stages of work on the teaching/learning trajectory on whole numbers for the higher primary grades. In the meantime it has been changed somewhat.

Student work as a mirror of education
How this problem is solved depends on how the students have been taught. In this way the problem can be seen as a mirror of the teaching. You get back what you have given the students. In the study in which the student work shown in Figure 19 was collected, almost a half of the fourth and fifth graders applied a column arithmetic procedure and less than one-third of the students chose a global estimation (see Treffers, Streefland and De Moor, 1996). Other studies of such problems revealed a strong connection with the type of education and different textbook series yielded different results (Treffers, 1999). The students' work makes it clear that there is still work to be done in the Netherlands, despite the good results found in the TIMSS study (see Mullis et al., 1997).

6 To conclude

The last example brings this guided tour to its end. It is hoped that the Dutch landscape of mathematics education is no longer the terra incognita that it might once have been before the reader started the tour. Dutch mathematics educators obviously have a special relation with landscapes. It was Freudenthal (1991) who called the last chapter in his last book 'The landscape of mathematics education.' This chapter probably inspired Treffers when he adapted a well-known poem by the famous Dutch poet Marsman to summarize mathematics education in Dutch primary schools.[27]

Thinking of Holland

Thinking of Holland
I see wide rivers
winding lazily through
endless low countryside
like rows of empty number lines
striping the horizon
I see multi-base
arithmetic blocks
low and lost
in the immense open space
and throughout the land
mathematics of a realistic brand

After
H. Marsman's 'Denkend aan Holland'

Adapted by
A. Treffers, 1996

[27] This adapted poem was presented by Adri Treffers at the start of an international meeting on mathematics education held at Leiden University, 14-15 December 1996.

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In for mathematics education in the Netherlands which de- addition to informing an international audience about the Dutch scribe the intended content. The term â€œcontentâ€ should be standards and curricula, we include some critical reflections on interpreted in a broad sense here.Â Mathematics lessons should give students the â€œguidedâ€ Alongside primary and secondary education there is opportunity to â€œre-inventâ€ mathematics by doing it.