Pauling 1932 *J. Amer. Chem. Soc.* **54**, 3570–82.
Pauling and Sherman 1933 *J. Chem. Phys.* **1**, 606–17.
Price 1936*a* *J. Chem. Phys.* **4**, 147–53.
— 1936*b* *J. Chem. Phys.* **4**, 539–49.
Price and Collins 1935 *Phys. Rev.* **48**, 714–19.
Price and Wood 1935 *J. Chem. Phys.* **3**, 439–44.
Rathenau 1933 *Z. Phys.* **87**, 32–56.
Sklar 1937 *J. Chem. Phys.* **5**, 669–81.
Smyth and Blewett 1934 *Phys. Rev.* **46**, 276–7.
Smyth and Mueller 1933 *Phys. Rev.* **43**, 121–2.
Watson and Parker 1931*a* *Phys. Rev.* **37**, 1013 (A).
— — 1931*b* *Phys. Rev.* **37**, 1484–92.
Wilson 1929 *Astrophys. J.* **69**, 34–42.

A suggestion for unifying quantum theory and relativity

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Introduction

There seems to be a general conviction that the difficulties of our present theory of ultimate particles and nuclear phenomena (the infinite values of the self energy, the zero energy and other quantities) are connected with the problem of merging quantum theory and relativity into a consistent unit. Eddington's book, "Relativity of the Proton and the Electron", is an expression of this tendency; but his attempt to link the properties of the smallest particles to those of the whole universe contradicts strongly my physical intuition. Therefore I have considered the question whether there may exist other possibilities of unifying quantum theory and the principle of general invariance, which seems to me the essential thing, as gravitation by its order of magnitude is a molar effect and applies only to masses in bulk, not to the ultimate particles. I present here an idea which seems to be attractive by its simplicity and may lead to a satisfactory theory.

1. Reciprocity

The motion of a free particle in quantum theory is represented by a plane wave

$$A \exp\left[\frac{i}{\hbar} p_k x^k\right],$$

where x^1, x^2, x^3, x^4 are the co-ordinates of space-time x, y, z, ct, and p_1, p_2, p_3, p_4 the components of momentum-energy p_x, p_y, p_z, E. The expression is completely symmetric in the two 4-vectors x and p. The transformation theory of quantum mechanics extends this "reciprocity" systematically. In a representation of the operators x^k, p_k in the Hilbert space for which the x^k are diagonal (δ-functions), the p_k are given by $\frac{\hbar}{i}\frac{\partial}{\partial x_k}$; and vice versa, if the p_k are diagonal the x^k are given by $-\frac{\hbar}{i}\frac{\partial}{\partial p_k}$. Any wave equation in the x-space can be transformed into another equation in the p-space, by help of the transformation

$$\phi(p) = \int \psi(x) \exp\left[\frac{i}{\hbar} p_k x^k\right] dx.$$

This reciprocity* can be extended also to the case of particles subject to external forces where the waves are not plane.

But there is a break in the reciprocal treatment when the principles of general relativity are applied. This theory has its origin in astronomical questions connected with the law of gravitation, and is founded on the conception of classical mechanics where the motion of a mass particle is represented not by a wave function, but by an orbit. The fundamental notion is the four-dimensional line element

$$ds^2 = g_{kl} dx^k dx^l, \tag{1}$$

the coefficients g_{kl} of which form the metric tensor.

It occurred to me that the principle of reciprocity would lead to the consideration of a line element in the p-space

$$d\sigma^2 = \gamma^{kl} dp_k dp_l \tag{2}$$

defining a metric in this space, but one which is not directly connected with the metric tensor g_{kl} in the x-space. If classical mechanics were valid throughout, this assumption would of course be impossible; for then p^k would be equal to $\mu \dot{x}^k$, where μ is the rest mass and the dot means differentiation with respect to proper time; therefore the transformation laws of the vector p would be completely determined by that of the vector x, and it would not be admissible to assume an independent absolute quadric for the determination of the metric in the p-space. But the real laws of nature are those of quantum theory. The classical conceptions refer only to a limiting case,

* The word "reciprocity" is chosen because it is already generally used in the lattice theory of crystals where the motion of the particle is described in the p-space with help of the "reciprocal lattice".

namely, that which is apt to describe the motion of molar bodies in space-time. It is characterized by the condition that energy and momentum of the quanta involved ($h\nu$ and h/λ) are extremely small (as compared with $h\nu_0$ and h/λ_0, where $\lambda_0 = c/\nu_0$ is the Compton wave length), whereas space and time are unlimited. There is another possibility of going over to a limit, namely, the case where we have to do with very small regions of space and time (as compared with λ_0 and $1/\nu_0$), but with unrestricted amounts of energy and momentum. This is the domain of ultimate particles and nuclear phenomena. It seems to me unjustified to assume that these two reciprocal limiting cases should be subject to the same metric, based on the line element in the x-space. I suggest that the conception of a metric is inapplicable for those phenomena in which x-space and p-space are involved simultaneously with about equal weight; it is only valid for the two limiting cases, for molar processes in the x-space, and for nuclear processes in the p-space. I have the impression that this assumption does not contradict any known fact. We have learned that the simultaneous measurement of a co-ordinate x^k and a momentum p_k are restricted by the uncertainty laws (which, by the way, conform to the principle of reciprocity, as they contain the x^k and p_k symmetrically). They should provide for the freedom necessary to have different and widely independent metrics for the two limiting cases, which we shall call, for sake of brevity, the molar and the nuclear world.

2. The differential equations for the metric tensors

In Einstein's theory of gravitation the metric tensor g_{kl} has to satisfy differential equations which connect the curvature tensor R_{kl} of space-time with the tensor energy-density T_{kl} of matter (including electromagnetic field). The most general form of these equations is

$$R_{kl} - (\tfrac{1}{2}R + \lambda)\, g_{kl} = -\kappa T_{kl}, \tag{3}$$

where κ is Einstein's gravitational constant and λ the cosmological constant. It is well known that these equations have a static solution corresponding to a closed (hyperspheric) world filled with matter of uniform density. Therefore there exists an upper limit for the distance between two points, given by the radius a of the universe.

Let us transfer this consideration to the p-space. For this purpose we have to define its curvature tensor P^{kl} in exactly the same way as the R_{kl} in the x-space. Further, we have to introduce quantities T^{kl} depending on the presence of matter. The meaning of these becomes clear if we remember that in the x-space the integrals $\int T_{k4}\,dx\,dy\,dz$ are momentum and energy of the system considered; analogously the integrals $\int \mathrm{T}^{k4}\,dp_x\,dp_y\,dp_z$ must be interpreted as space co-ordinates and time value of the system. We have, therefore, in accordance with our general considerations, to attribute to the whole system one single point in space (which may move in time); spatial specifications of the parts of the system are meaningless, whereas we have full freedom to study the energetic processes of the parts.

This seems to be a proper way of dealing with internuclear processes. As far as I can see the existing theories of the nuclei are of this type. For the description of the fundamental properties of a nucleus it seems to be unnecessary to specify carefully the law of interaction between its constituent particles; any function of the distance will do, if only the total range of action and the dissociation energy are properly chosen. The fully developed theory should, of course, modify the extreme p-standpoint and allow some statements on spatial properties of nuclei in accordance with the uncertainty rules.

The differential equation for the metric tensor γ^{kl} in the p-space will have the same form as that in the x-space, namely

$$\mathrm{P}^{kl} - (\tfrac{1}{2}\mathrm{P} + \lambda')\,\gamma^{kl} = -\kappa'\mathrm{T}^{kl}, \tag{4}$$

where λ' and κ' are constants. Whether these nuclear constants are connected with the corresponding molar constants λ, κ cannot be decided yet.

3. Hyperspherical momentum space

The equations (4) will have a solution corresponding to a closed (hyperspheric) momentum space (p_x, p_y, p_z), independent of E. Therefore we are led to the conclusion that for systems of some kind there is an *upper limit for momenta*,* determined by the radius b of the hypersphere. The systems to which this idea is applicable must be *energetically closed*; it certainly does not apply to every system, as we know the existence of particles with any amount of momentum and energy (cosmic rays).

This result is of great importance, as it removes immediately the infinities which are the dark points of the present theories. The hypersphere can be

* This assumption has already been made, but without any relativistic foundation, by M. Born and G. Rumer (1931). See also G. Wataghin (1934) and A. March (1937). Quite a different way of avoiding the infinite self-energy has been suggested by G. Wentzel (1933, 1934).

written by help of a parameter u, having the character of a momentum,

$$p_x^2+p_y^2+p_z^2+u^2 = b^2; \tag{5}$$

from this we get

$$u = (b^2-p^2)^{\frac{1}{2}}, \quad u\,du = -(p_x dp_x+p_y dp_y+p_z dp_z).$$

The line element of the p-space we get by eliminating u and du from

$$d\sigma^2 = dE^2-(dp_x^2+dp_y^2+dp_z^2+du^2)$$

in the form

$$d\sigma^2 = dE^2-\Big\{dp_x^2\Big(1+\frac{p_x^2}{b^2-p^2}\Big)+dp_y^2\Big(1+\frac{p_y^2}{b^2-p^2}\Big)+dp_z^2\Big(1+\frac{p_z^2}{b^2-p^2}\Big)$$
$$+2dp_y dp_z\frac{p_y p_z}{b^2-p^2}+2dp_z dp_x\frac{p_z p_x}{b^2-p^2}+2dp_x dp_y\frac{p_x p_y}{b^2-p^2}\Big\}. \tag{6}$$

We omit the well-known proof that the γ^{kl} defined by (6) are solutions of the differential equations (4) if b is suitably chosen as a function of λ', κ'.

The three-dimensional volume element is given by

$$d\Omega = \sqrt{(-\gamma)}\,dp_x dp_y dp_z,$$

where

$$-\gamma = \begin{vmatrix} 1+\frac{p_x^2}{b^2-p^2} & \frac{p_x p_y}{b^2-p^2} & \frac{p_x p_z}{b^2-p^2} \\ \frac{p_y p_x}{b^2-p^2} & 1+\frac{p_y^2}{b^2-p^2} & \frac{p_y p_z}{b^2-p^2} \\ \frac{p_z p_x}{b^2-p^2} & \frac{p_z p_y}{b^2-p^2} & 1+\frac{p_z^2}{b^2-p^2} \end{vmatrix} = 1+\frac{p^2}{b^2-p^2} = \frac{b^2}{b^2-p^2}.$$

This shows that b is the upper limit of p. We get

$$d\Omega = \frac{dp_x dp_y dp_z}{\sqrt{(1-p^2/b^2)}}. \tag{7}$$

This simple result admits of some important applications. For if we have to do with a system of independent particles, the fundamental law of quantum statistics gives the *number of quantum states of weight g in a spatial volume V and a momentum element $d\Omega$*

$$dn = g\frac{V}{h^3}d\Omega = g\frac{V}{h^3}\frac{dp_x dp_y dp_z}{\sqrt{(1-p^2/b^2)}}, \tag{8}$$

The appearance of the square root indicates deviations from the classical laws; it removes, as stated above, the disturbing infinities. We shall show this for a few examples connected with quantum electrodynamics.

The *total number of quantum states* in V is finite, namely,

$$n = \int dn = g\frac{V}{h^3}\iiint \frac{dp_x dp_y dp_z}{\sqrt{(1-p^2/b^2)}} = g\frac{4\pi V b^3}{h^3}\int_0^1 \frac{\xi^2 d\xi}{\sqrt{(1-\xi^2)}} = g\frac{\pi^2 V b^3}{h^3}. \tag{9}$$

The important question arises whether the constant b is universal, or characteristic for each energetically closed system. I do not think this question can be answered in the present preliminary state of the theory. For the sake of argument I shall assume in the following examples of application that the value of b is always the same.

4. Application to quantum electrodynamics

We use the form of quantized electrodynamics given by Fermi (1932). He writes the Hamiltonian for a system of electrons in an electromagnetic field contained in the volume V, as the operator

$$H = H_e + H_r + H_i + \frac{1}{\pi V}\sum_s \frac{h^2}{p_s^2}(\sum_k e_k \cos \Gamma_{sk})^2; \tag{10}$$

here

$$H_e = -\sum_k \{c(\vec{\alpha}_k \vec{p}_k) + m_k c^2 \beta_k\} \tag{11}$$

is the Dirac Hamiltonian for the electrons ($k = 1, 2, \ldots$) with rest mass m_k;

$$H_r = \sum_s \{\tfrac{1}{2}(p_{s1}^2 + p_{s2}^2) + 2\pi^2\nu_s^2(q_{s1}^2 + q_{s2}^2)\} \tag{12}$$

the energy of the oscillators representing the radiation field;

$$H_i = \sum_k e_k c \sqrt{\frac{8\pi}{V}} \sum_s \{\vec{\alpha}_k(\vec{A}_{s1} q_{s1} + \vec{A}_{s2} q_{s2})\} \sin \Gamma_{sk} \tag{13}$$

the interaction energy between electrons and radiation, where $\vec{A}_{s1}$, $\vec{A}_{s2}$ are two unit vectors orthogonal to one another and to the direction of propagation of the wave s, which, at the place of the electron k, has the phase

$$\Gamma_{sk} = \frac{2\pi}{h}(p_{sx} x_k + p_{sy} y_k + p_{sz} z_k) + \delta_s. \tag{14}$$

This theory represents the facts of radiation marvellously, but it involves some infinities. The simplest of these are:

(1) The zero energy of radiation contained in H_r; for the stationary states one has

$$H_r = \sum_s h\nu_s(n_s + \tfrac{1}{2}). \tag{15}$$

(2) The Coulomb self-energy of the electrons contained in the term given explicitly in (9), namely,

$$H_c = \frac{1}{\pi V}\sum_s \frac{h^2}{p_s^2}\left(\sum_k e_k \cos\Gamma_{sk}\right)^2. \tag{16}$$

All these formulae may possibly need modifications as a consequence of the p-metric. But I do not expect these alterations will be essential, and I shall suppose here that the only effect of the p-metric is that on the counting of quantum states.

With help of (8), where $g = 2$ corresponding to the two directions of polarization, the zero energy of radiation becomes

$$E_0 = H_r^0 = \sum_s \tfrac{1}{2}h\nu_s = \frac{c}{2}\sum_s p_s = \frac{8\pi V_c}{2h^3}\int_0^b \frac{p^3\,dp}{\sqrt{(1-p^2/b^2)}}$$
$$= \frac{4\pi c V b^4}{h^3}\int_0^1 \frac{\xi^3\,d\xi}{\sqrt{(1-\xi^2)}} = \frac{8\pi}{3}\frac{cVb^4}{h^3},$$

or with help of (9)

$$E_0 = \frac{4}{3\pi} cbn, \tag{17}$$

which has, in fact, the dimension of energy. The Coulomb interaction (16) can be written

$$H_c = \frac{h^2}{\pi V}\sum_{k,l} e_k e_l R_{kl}, \tag{18}$$

with

$$R_{kl} = \sum_s \frac{\cos\Gamma_{sk}\cos\Gamma_{sl}}{p_s^2} = \frac{V}{h^3}\iiint \cos\Gamma_k \cos\Gamma_l \frac{dp_x\,dp_y\,dp_z}{p^2\sqrt{(1-p^2/b^2)}}, \tag{19}$$

where the weight g has to be taken equal to 1 (longitudinal waves).

We average over all phases δ_s and introduce the cosine γ of the angle between the vectors $\vec{p}$ and $\vec{r}_{kl} = \vec{r}_k - \vec{r}_l$. Then

$$R_{kl} = \frac{2\pi V}{2h^3}\int_0^b\int_{-1}^1 \cos\left(\frac{2\pi}{h} p\gamma r_{kl}\right)\frac{dp\,d\gamma}{\sqrt{(1-p^2/b^2)}}$$
$$= \frac{V}{h^2}\int_0^b \frac{\sin\left(\dfrac{2\pi}{h} p r_{kl}\right)}{p r_{kl}}\frac{dp}{\sqrt{(1-p^2/b^2)}}.$$

If we introduce the function

$$f(x) = \frac{2}{\pi}\int_0^1 \frac{\sin(\xi x)}{\xi}\frac{d\xi}{\sqrt{(1-\xi^2)}} = \int_0^x J_0(y)\,dy, \tag{20}$$

where $J_0(y)$ is the Bessel function, we get

$$R_{kl} = \frac{\pi V}{2h^2}\frac{1}{r_{kl}} f\left(2\frac{r_{kl}}{r_0}\right), \tag{21}$$

with

$$r_0 = \frac{h}{\pi b}. \tag{22}$$

Substituting (21) in (18) we get the modified Coulomb law

$$H_c = \tfrac{1}{2}\sum_{k,l} \frac{e_k e_l}{r_{kl}} f\left(2\frac{r_{kl}}{r_0}\right). \tag{23}$$

From the definition (20) one finds easily

$$\underset{x\to\infty}{\mathrm{L}}\ f(x) = 1, \quad \underset{x\to 0}{\mathrm{L}}\ \frac{1}{x} f(x) = 1. \tag{24}$$

Therefore we have the classical Coulomb law for $r_{kl} \gg r_0$, and r_0 determines evidently the "dimensions" of the electron. One finds its precise meaning by calculating the self-energy terms in (23), taking $k = 1$, namely,

$$\frac{e^2}{r_0} = mc^2, \tag{25}$$

where we have introduced the mass m of the electron. Therefore r_0 is the classical radius of the electron, $r_0 = e^2/mc^2 = 2{\cdot}80 \times 10^{-13}$ cm., and we get from (22)

$$b = \frac{h}{\pi r_0} = 7{\cdot}43 \times 10^{-15}\,\text{g. cm. sec.}^{-1}. \tag{26}$$

As the terms H_c account for the inertia of the electrons one would be inclined to omit the mass terms in the Dirac Hamiltonian H_e (11); but these appear there multiplied by the spin operator β. This shows that a complete explanation of mass as an electromagnetic phenomenon requires a deeper understanding of the relation of the spin and the electromagnetic field. We shall not go into this question.

Introducing r_0 from (22) instead of b in (9) and (17), one gets

$$n = n_0 V, \quad n_0 = \frac{2}{\pi r_0^3} = 2{\cdot}90 \times 10^{37}\,\text{cm.}^{-3}; \tag{27}$$

$$E_0 = \epsilon_0 n = \epsilon_0 n_0 V, \quad \epsilon_0 = \frac{4}{3\pi^2}\frac{ch}{r_0} = \frac{8}{3\pi}\frac{hc}{2\pi e^2}\frac{e^2}{r_0} = \frac{8}{3\pi}137\,mc^2$$

$$= 117\,mc^2 = 9{\cdot}49 \times 10^{-5}\,\text{erg} = 5{\cdot}97 \times 10^7\,\text{e-volts};$$

$$\epsilon_0 n_0 = \frac{16}{3\pi^2}\frac{h^2}{2\pi e^2}\left(\frac{e}{r_0^2}\right)^2 = \frac{16}{3\pi^2}137\left(\frac{e}{r_0^2}\right)^2 = 2{\cdot}75 \times 10^{33}\,\text{erg cm.}^{-1}. \tag{28}$$

This shows that the zero energy per quantum oscillator, ϵ_0, is $8/3\pi \times 137$ times the rest energy of the electron, mc^2; and that the density of the zero energy of radiation, $E_0/V = \epsilon_0 n_0$, is $128/3\pi \times 137$ times the electrostatic energy density $\mathscr{E}^2/8\pi$, where $\mathscr{E} = e/r_0^2$ is the electric field at the "surface" of the electron.

The numerical values should be considered as preliminary, since the electromagnetic or transverse self-energy which arises from the term H_i, (13), has to be added. Dirac's single electron theory gives, according to Heitler (1936), for this transverse self-energy an expression which would lead to a value about 2×137 times as large as the electrostatic one. But it has been shown by Weisskopf (1934) and Kemmer (1935) that the hole theory of the electron leads to another expression which gives a value of the same order as the electrostatic one, differing only by a numerical factor $\frac{1}{2}$. In connexion with this question it should be considered whether the value of b for the longitudinal waves (electrostatic terms) and the transversal waves (electromagnetic terms) is necessarily identical.

5. Heat radiation

We can now apply formula (8) to the excited states of the radiation field. As the partition function of an oscillator with the energy $h\nu n$ is

$$Q_\nu = \sum_{n=0}^{\infty} e^{-h\nu n/kT} = \frac{1}{1-e^{-h\nu/kT}}, \tag{29}$$

we find for the free energy of the radiation field

$$F = -kT\sum_\nu \ln Q_\nu = kT\sum_\nu \ln(1-e^{-h\nu/kT}), \tag{30}$$

and if we assume that $p = h\nu/c$, we have with the help of (8)

$$F = \frac{8\pi kT}{c^3} V \int_0^{1/\tau} \ln(1-e^{-h\nu/kT}) \frac{\nu^2 d\nu}{\sqrt{(1-(\nu\tau)^2)}}, \tag{31}$$

with

$$\tau = \frac{h}{bc} = \frac{\pi r_0}{c} = 2{\cdot}94 \times 10^{-23}\,\text{sec.}; \tag{32}$$

here we have assumed that b is the same as determined above; then τ is the time which light needs to travel the distance πr_0. The entropy is

$$S = -\frac{\partial F}{\partial T} = -\frac{F}{T} + \frac{8\pi h}{c^3} V \frac{1}{T} \int_0^{1/\tau} \frac{\nu^3 d\nu}{(e^{h\nu/kT}-1)\sqrt{(1-(\nu\tau)^2)}} \tag{33}$$

and the energy
$$U = F + TS = V\int_0^{1/\tau} u(v, T)\,dv \tag{34}$$

with
$$u(\nu, T) = \frac{8\pi h}{c^3}\frac{\nu^3}{(e^{h\nu/kT}-1)\sqrt{(1-(\nu\tau)^2)}}. \tag{35}$$

This is the modified Planck formula for the density of radiation. The radiation pressure is given by

$$\begin{aligned} P = -\frac{\partial F}{\partial V} = -\frac{F}{V} &= -\frac{8\pi kT}{c^3}\int_0^{1/\tau} \ln(1-e^{-h\nu/kT})\frac{\nu^2 d\nu}{\sqrt{(1-(\nu\tau)^2)}} \\ &= -n_0 kT\frac{4}{\pi}\int_0^1 \ln(1-e^{-\Theta_0\xi/T})\frac{\xi^2 d\xi}{\sqrt{(1-\xi^2)}}, \end{aligned} \tag{36}$$

where

$$\left.\begin{aligned} k\Theta_0 = \frac{h}{\tau} = bc = \frac{hc}{\pi r_0} = \frac{3\pi}{4}\epsilon_0 = 2{\cdot}23\times 10^{-4}\,\text{erg} = 1{\cdot}41\times 10^{8}\,\text{e-volts}, \\ \Theta_0 = 1{\cdot}63\times 10^{12}\,\text{degrees}. \end{aligned}\right\} \tag{37}$$

These numerical values should be considered with reserve, as mentioned above.

The total energy density can be written

$$\begin{aligned} u(T) = \int_0^{1/\tau} u(\nu, T)\,d\nu &= n_0 k\Theta_0\frac{4}{\pi}\int_0^1 \frac{\xi^3 d\xi}{(e^{\Theta_0\xi/T}-1)\sqrt{(1-\xi^2)}} \\ &= n_0 k\Theta_0\frac{4}{\pi}\int_0^{\pi/2} \frac{\sin^3\phi\,d\phi}{e^{\Theta_0\sin\phi/T}-1}. \end{aligned} \tag{38}$$

The quantity $u-3P$ which vanishes in the classical theory differs here from zero, namely

$$u(T)-3P(T) = n_0 kT\frac{4}{\pi}\int_0^1 \frac{\xi^2 d\xi}{\sqrt{(1-\xi^2)}}\left\{\frac{\xi\Theta_0/T}{e^{\xi\Theta_0/T}-1}+3\ln(1-e^{-\Theta_0\xi/T})\right\}. \tag{39}$$

This vanishes for $T\to 0$, but has for $T \gg \Theta_0$ the value $n_0 kT$, corresponding to a kind of saturation as if each degree of freedom of the vacuum had acquired the equipartition value kT of energy. As a matter of fact, the formula (38) for the energy is formally similar to that of a (one-dimensional) crystal lattice, as studied a long time ago by v. Kármán and myself (1912, 1913). One has for $T \ll \Theta_0$ the Stephan-Boltzmann law

$$u(T) = aT^4, \quad a = \frac{8\pi^5}{15}\frac{n_0 k}{\Theta_0^3} = \frac{8\pi^5 k^4}{15 c^3 h^3}, \tag{40}$$

whereas for $T \gg \Theta_0$
$$u(T) = n_0 kT; \tag{41}$$
in this region the vacuum follows the law of Dulong-Petit.

These results show that the radiation pressure cannot be considered as the transfer of momentum $p = h\nu/c$. This holds only for temperatures low compared with Θ_0; for higher temperatures the pressure has more the character of the internal pressure of a vibrating crystal lattice. It follows that Maxwell's equations cannot hold for high-frequency waves, but have to be modified in such a way that the relation of the pressure of light to the density of energy is consistent with (39). But as Fermi's formulae from which we started are nothing but the quantized Maxwell's equations there may possibly be deeper alterations necessary affecting all the formulae of this section.

6. Kinetic theory of gases

There are also deviations from the accepted laws of the kinetic theory of gases. The partition function per molecule becomes*

$$Q = \frac{V}{h^3}\iiint e^{-p^2/2MkT}\frac{dp_x dp_y dp_z}{\sqrt{(1-p^2/b^2)}}$$

or
$$Q = Vn_0\frac{2}{\pi}\int_0^1 e^{-\Theta\xi^2/T}\frac{\xi^2 d\xi}{\sqrt{(1-\xi^2)}} = Vn_0\tfrac{1}{2}e^{-\Theta/2T}\left\{J_0\left(\frac{i\Theta}{2T}\right)+iJ_1\left(\frac{i\Theta}{2T}\right)\right\}, \tag{42}$$

where Θ can be expressed by the characteristic temperature of the vacuum:

$$\Theta = \frac{b^2}{2Mk} = \frac{m}{M}\frac{hc}{2\pi e^2}\Theta_0 = \frac{137}{1845\mu}\Theta_0 = \frac{1{\cdot}21\times10^{11}}{\mu}\text{ degree.} \tag{43}$$

μ is the molecular weight relative to the H-atom. One has for $T \ll \Theta$ the usual formula

$$Q = V\frac{1}{2\sqrt{\pi}}n_0\left(\frac{T}{\Theta}\right)^{\frac{3}{2}} = V\left(\frac{2\pi MkT}{h^2}\right)^{\frac{3}{2}}, \tag{44}$$

but for $T \gg \Theta$

$$Q = \tfrac{1}{2}Vn_0\left\{1-\frac{3}{4}\frac{\Theta}{T}+\frac{5}{16}\left(\frac{\Theta}{T}\right)^2-\frac{35}{384}\left(\frac{\Theta}{T}\right)^3+\frac{21}{1024}\left(\frac{\Theta}{T}\right)^4-\dots\right\}. \tag{45}$$

This high temperature degeneration has no influence on the equation of state, but on the specific heat. It may play a role in the theory of the constitution of stars, and on the constitution of nuclei as well, as these, according to Bohr and Kalckar (1937), can be treated by thermodynamical methods.

* This problem can also be treated relativistically quite easily; however, the result depends essentially on what assumption one makes about the radius b.

The molar heat for high temperatures, $T \gg \Theta$, is

$$c_v = \frac{R}{16}\frac{\Theta^2}{T^2} + \ldots, \tag{46}$$

and goes to zero for $T \to \infty$.

Conclusion

A consequence of the assumption of a finite size of a system in the p-space is the existence of a set of proper functions $\psi_n(p)$, where the index n refers to proper values of some functions of the space co-ordinates. This means that our theory leads to a kind of granular or lattice structure of space without introducing such a strange assumption a priori.

The suggestions made in this paper contain an ample programme for further investigation; the most important question seems to me the generalization of the idea of the metric tensor and of the equations determining it, for that intermediate region where classical methods neither in the x-space nor in the p-space are applicable.

Summary

The fact that the fundamental laws of quantum mechanics are symmetrical in space-time x^k and momentum-energy p_k can be generalized to a "principle of reciprocity" according to which the x-space and the p-space are subject to geometrical laws of the same structure, namely a Riemannian metric. In analogy with Einstein's closed x-world one has to assume that energetically closed systems (as elementary particles, nuclei) must be described by help of a hyperspherical p-space. A consequence of this assumption is a modification of the formula for the number of quantum states in an element of the p-space. The application of this formula to quantum electrodynamics leads to a finite zero energy of the vacuum, a finite self-energy of the electron, etc. Deviations from Planck's law and the Stephan-Boltzmann law of radiation, and the calorie properties of gases are predicted for very high temperatures.*

References

Bohr, N. and Kalckar, F. 1937 *Kgl. Danske Vidensk. Selskab.*, Math.-fys. Med., **14**, 10. See further V. Weisskopf, *Phys. Rev.*, **52**, 295 (1937).

Born, M. and Kármán, Th. v. 1912 *Phys. Z.* **13**, 297.

—— 1913 *Phys. Z.* **14**, 65.

* [*Note added in proof.*] The application of this theory to nuclei leads to results confirming the assumptions. Cf. *Nature*, **141**, 327 (1938).

Born, M. and Rumer, G. 1931 *Z. Phys.* **69**, 141.
Fermi, E. 1932 "Quantum theory of Radiation", *Rev. Mod. Phys.* **4**, 87; see especially formula (166), p. 130.
Heitler, W. 1936 "The Quantum theory of Radiation", Chap. III, § 18, p. 183. (Oxford University Press.)
Kemmer, N. 1935 *Ann. Phys., Lpz.* (5), **22**, 674.
March, A. 1937 *Z. Phys.* **104**, 93, 161; **105**, 620; **106**, 49, 291, 532; **108**, 128.
Wataghin, G. 1934 *Z. Phys.* **88**, 92.
Weisskopf, V. 1934 *Z. Phys.* **89**, 27; **90**, 817.
Wentzel, G. 1933 *Z. Phys.* **86**, 479, 635.
— 1934 *Z. Phys.* **87**, 726.

Hyperfine structure, Zeeman effect and isotope shift in the resonance lines of potassium

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[Plate 5]

Introduction

In an earlier work (Jackson and Kuhn 1935, 1936) the hyperfine structure in the resonance lines of the abundant isotope 39 of potassium was observed by the method of absorption in an atomic beam; but no intensity measurements were made. Qualitatively, the short wave-length component appeared to be the stronger, which led to the assumption of a negative magnetic moment. Magnetic deflexion experiments (Millman 1935; Fox and Rabi 1935), though in accurate agreement as regards the width of splitting, gave a positive magnetic moment. The absorption experiments were therefore repeated under conditions which excluded overlapping of neighbouring orders of the interferometer spectrum and thus permitted a quantitative determination of the intensities. This was achieved by using an etalon of 5 cm. length only (instead of 10 cm. in the old experiment) and by running the light source at low pressure of potassium. The measurements, the main results of which were published in a preliminary note (Jackson and Kuhn 1937*a*), gave an intensity ratio 1·45 of the hyperfine structure components, the long wave-length one being the stronger.

Therefore the combination of General Relativity and Quantum Physics will yield a Relativistic Quantum Theory of Gravity which will include Quantum as well as Relativistic effects to explain the nature of Gravity on medium scale objects. We need such a theory in order to get the Theory of Everything (T.O.E) as stated earlier but such a theory would also help us study Singularities. Since in a Singularity, the density of a body is so tremendous that Quantum Effects cannot be ignored whereas the gravitational force is so strong that General Relativity cannot be ignored too.Â The future research will show if this is a good path for unifying all four known forces in a quantum mechanical underlying framework. share|cite|improve this answer. Born M., A suggestion for unifying quantum theory and relativity, Proc. R. Soc. Lond. Ser. A 165 (1938), 291-303. de Sousa Gerbert P., On spin and (quantum) gravity in 2+1 dimensions, Nuclear Phys. B 346 (1990), 440-472.Â Freidel L., Livine E.R., 3D quantum gravity and effective noncommutative quantum field theory, Phys. Rev. Lett. 96 (2006), 221301, 4 pages, hep-th/0512113. Freidel L., Livine E.R., Ponzano-Regge model revisited. III. Feynman diagrams and effective field theory, Classical Quantum Gravity 23 (2006), 2021-2061, hep-th/0502106. Quantum theories currently only deal with 3 of the four known forces in nature, the electromagnetic, strong nuclear force and the weak nuclear force and at present cannot deal with gravitational forces. Symmetry has been a very strong guiding principle in the development of the various quantum theories and the incorporation of the various forces as they became known and understood. That gravity does not fit pattren which has allowed a selfconsistent theories incorporating the three other forces to be developed is highly unsatisfying .Â I think when we answer the question: what is the purpose of unifying General Relativity and Quantum Theory? It is not enough to say 'we want to solve this theoretical problem!'