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Proving some geometric inequalities by using complex numbers

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Abstract

Let ABC be a triangle and let R and r be its circumradius and inradius, respectively. One of the most important result in Triangle Geometry is Euler's inequality $R \geq 2r$. There are many proofs for this inequality (geometric, trigonometric, analytic etc.). We refer to the books [3] and [4] for some useful discussions on this inequality.

In this note we will give other proofs by using complex numbers. The method of complex numbers in Geometry is a powerful technique. For other applications we refer to our new book [2].

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Theorem 1. *Let P be an arbitrary point in the plane of triangle ABC. Then*

$$\alpha PB \cdot PC + \beta PC \cdot PA + \gamma PA \cdot PB \geq \alpha\beta\gamma,$$

where α, β, γ are the side lengths of triangle ABC.

Proof. Let us consider the origin of the complex plane at P and let a, b, c be the affixes of vertices of triangle ABC. From the algebraic identity

$$\frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-c)(b-a)} + \frac{ab}{(c-a)(c-b)} = 1 \tag{1}$$

by passing to moduli, it follows that

$$\frac{|b||c|}{|a-b||a-c|} + \frac{|c||a|}{|b-c||b-a|} + \frac{|a||b|}{|c-a||c-b|} \geq 1. \tag{2}$$

Taking into account that $|a| = PA$, $|b| = PB$, $|c| = PC$ and $|b-c| = \alpha$, $|c-a| = \beta$, $|a-b| = \gamma$, (2) is equivalent to

$$\frac{PB \cdot PC}{\beta\gamma} + \frac{PC \cdot PA}{\gamma\alpha} + \frac{PA \cdot PB}{\alpha\beta} \geq 1,$$

i.e. the desired inequality.

Remarks. 1) If P is the circumcenter O of triangle ABC we can derive Euler's inequality $R \geq 2r$. Indeed, in this case the inequality is equivalent to $R^2(\alpha+\beta+\gamma) \geq \alpha\beta\gamma$. Therefore we can write

$$R^2 \geq \frac{\alpha\beta\gamma}{\alpha+\beta+\gamma} = \frac{\alpha\beta\gamma}{2s} = \frac{4R}{2s} \cdot \frac{\alpha\beta\gamma}{4R} = 2R \cdot \frac{area[ABC]}{s} = 2Rr,$$

hence $R \geq 2r$.

2) We can obtain the inequality

$$R^2(\alpha+\beta+\gamma) \geq \alpha\beta\gamma \tag{3}$$

by a different argument, but also by using complex numbers. This alternative proof is given in our book [1]. Indeed, with the notations in the proof of Theorem 1, we have the identity

$$a^2(b-c) + b^2(c-a) + c^2(a-b) = (a-b)(b-c)(c-a). \tag{4}$$

Passing to moduli and using the well-known triangle inequality, we obtain

$$|a-b||b-c||c-a| \leq |a|^2|b-c| + |b|^2|c-a| + |c|^2|a-b|. \tag{5}$$

Suppose that the circumcenter O of triangle ABC is the origin of the complex plane. Then $|a| = |b| = |c| = R$ and (5) is equivalent to inequality (3).

3) If P is the centroid G of triangle ABC, we derive the following inequality involving the medians $m_\alpha, m_\beta, m_\gamma$:

$$\frac{m_\alpha m_\beta}{\alpha\beta} + \frac{m_\beta m_\gamma}{\beta\gamma} + \frac{m_\gamma m_\alpha}{\gamma\alpha} \geq \frac{9}{4},$$

with equality if and only if triangle ABC is equilateral.

Some Olympiad-calliber problems are directly connected to the result contained in Theorem 1. The first such problem deals with the case of equality when triangle ABC is acute-angled.

Problem 1. *Let ABC be an acute-angled triangle and let P be a point in its interior. Prove that*

$$\alpha \cdot PB \cdot PC + \beta \cdot PC \cdot PA + \gamma \cdot PA \cdot PB = \alpha\beta\gamma,$$

if and only if P is the orthocenter of triangle ABC.

(1998 Chinese Mathematical Olympiad)

Solution. Let P be the origin of the complex plane and let a, b, c be the affixes of A, B, C, respectively. The relation in the problem is equivalent to

$$|ab(a-b)| + |bc(b-c)| + |ca(c-a)| = |(a-b)(b-c)(c-a)|.$$

Let

$$z_1 = \frac{ab}{(a-c)(b-c)}, \quad z_2 = \frac{bc}{(b-a)(c-a)}, \quad z_3 = \frac{ca}{(c-b)(a-b)}.$$

It follows that

$$|z_1| + |z_2| + |z_3| = 1 \quad \text{and} \quad z_1 + z_2 + z_3 = 1,$$

the latter from identity (1) in the previous problem.

We will prove that P is the orthocenter of triangle ABC if and only if z_1, z_2, z_3 are positive real numbers. Indeed, if P is the orthocenter, then, since the triangle ABC is acute-angled, it follows that P is in the interior of ABC. Hence there are positive real numbers r_1, r_2, r_3 such that

$$\frac{a}{b-c} = -r_1 i, \quad \frac{b}{c-a} = -r_2 i, \quad \frac{c}{a-b} = -r_3 i,$$

implying $z_1 = r_1 r_2 > 0$, $z_2 = r_2 r_3 > 0$, $z_3 = r_3 r_1 > 0$ and we are done. Conversely, suppose that z_1, z_2, z_3 are all positive real numbers. Because

$$-\frac{z_1 z_2}{z_3} = \left(\frac{b}{c-a}\right)^2, \quad -\frac{z_2 z_3}{z_1} = \left(\frac{c}{a-b}\right)^2, \quad -\frac{z_3 z_1}{z_2} = \left(\frac{a}{b-c}\right)^2$$

it follows that

$$\frac{a}{b-c}, \ \frac{b}{c-a}, \ \frac{c}{a-b}$$

are pure imaginary numbers, thus $AP \perp BC$ and $BP \perp CA$, showing that P is the orthocenter of triangle ABC.

Problem 2. *Let G be the centroid of triangle ABC and let R_1, R_2, R_3 be the circumradii of triangles GBC, GCA, GAB, respectively. Then*

$$R_1 + R_2 + R_3 \geq 3R,$$

where R is the circumradius of triangle ABC.

Solution. In Theorem 1, let P be the centroid G of triangle ABC. Then

$$\alpha \cdot GB \cdot GC + \beta \cdot GC \cdot GA + \gamma \cdot GA \cdot GB \geq \alpha\beta\gamma, \tag{6}$$

where α, β, γ are the side lengths of triangle ABC.

But

$$\alpha \cdot GB \cdot GC = 4R_1 \cdot area[GBC] = 4R_1 \cdot \frac{1}{3} area[ABC]$$

and the other two relations:

$$\beta \cdot GC \cdot GA = 4R_2 \cdot \frac{1}{3} area[ABC], \quad \gamma \cdot GA \cdot GB = 4R_3 \cdot \frac{1}{3} area[ABC].$$

Hence (6) is equivalent to

$$\frac{4}{3}(R_1 + R_2 + R_3) \cdot area[ABC] \geq 4R \cdot area[ABC],$$

i.e. $R_1 + R_2 + R_3 \geq 3R$, as desired.

Problem 3. *Let ABC be a triangle and let P be a point in its interior. Let R_1, R_2, R_3 be the radii of the circumcircles of triangles PBC, PCA, PAB, respectively. Lines PA, PB, PC intersect sides BC, CA, AB at A_1, B_1, C_1, respectively. Denote*

$$k_1 = \frac{PA_1}{AA_1}, \quad k_2 = \frac{PB_1}{BB_1}, \quad k_3 = \frac{PC_1}{CC_1}.$$

Prove that

$$k_1R_1 + k_2R_2 + k_3R_3 \geq R,$$

where R is the circumradius of triangle ABC.

(2004 Romanian IMO Team Selection Test)

Solution. Note that

$$k_1 = \frac{area[PBC]}{area[ABC]}, \quad k_2 = \frac{area[PCA]}{area[ABC]}, \quad k_3 = \frac{area[PAB]}{area[ABC]}.$$

But $area[ABC] = \dfrac{\alpha\beta\gamma}{4R}$ and $area[PBC] = \dfrac{\alpha \cdot PB \cdot PC}{4R_1}$. Other two similar relations for $area[PCA]$ and $area[PAB]$ hold.

The desired inequality is equivalent to

$$R\frac{\alpha \cdot PB \cdot PC}{\alpha\beta\gamma} + R\frac{\beta \cdot PC \cdot PA}{\alpha\beta\gamma} + R\frac{\gamma \cdot PA \cdot PB}{\alpha\beta\gamma} \geq R,$$

which reduces to the inequality in Theorem 1.

In the case when triangle ABC is acute-angled, from Problem 1 it follows that equality holds if and only if P is the orthocenter of ABC.

Theorem 2. *Let P be an arbitrary point in the plane of triangle ABC. Then*

(7) $$\alpha \cdot PA^2 + \beta \cdot PB^2 + \gamma \cdot PC^2 \geq \alpha\beta\gamma.$$

Proof. Let us consider the origin of the complex plane at the point P and let a, b, c be the affixes of the vertices of triangle ABC. The following identity is easy to verify:

$$\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} = 1. \tag{8}$$

By passing to moduli it follows that

$$1 = \left|\sum_{cyc} \frac{a^2}{(a-b)(a-c)}\right| \leq \sum_{cyc} \frac{|a|^2}{|a-b||a-c|}$$

Taking into account that $|a| = PA$, $|b| = PB$, $|c| = PC$ and $|b-c| = \alpha$, $|c-a| = \beta$, $|a-b| = \gamma$, the previous inequality is equivalent to (7).

Remarks. 1) If P is the circumcenter O of triangle ABC, then $PA = PB = PC = R$ and from (8) we derive again inequality (3), which is equivalent to Euler's inequality $R \geq 2r$.

2) If P is the centroid G of triangle ABC, then

$$PA^2 = \frac{1}{9}[2(\beta^2+\gamma^2) - \alpha^2], \quad PB^2 = \frac{1}{9}[2(\gamma^2+\alpha^2) - \beta^2],$$

$$PC^2 = \frac{1}{9}[2(\alpha^2+\beta^2) - \gamma^2]$$

and (7) is equivalent to

$$2\sum_{cyc}(\beta^2+\gamma^2) \geq 9\alpha\beta\gamma + \alpha^3 + \beta^3 + \gamma^3. \tag{9}$$

3) If P is the incenter I of triangle ABC, then

$$PA = \frac{r}{\sin\frac{A}{2}}, \quad PB = \frac{r}{\sin\frac{B}{2}}, \quad PC = \frac{r}{\sin\frac{C}{2}}$$

and is not difficult to see that we have equality in (7).

4) A different proof for (7), by using a variant of Lagrange's identity, is given in the book [4].

Theorem 3. *Let P be an arbitrary point in the plane of triangle ABC. Then*

$$\alpha \cdot PA^3 + \beta \cdot PB^3 + \gamma \cdot PC^3 \geq 3\alpha\beta\gamma PG, \tag{10}$$

where G is the centroid of triangle ABC.

Proof. The identity

$$x^3(y-z) + y^3(z-x) + z^3(x-y) = (x-y)(y-z)(z-x)(x+y+z) \tag{11}$$

holds for any complex numbers x, y, z. Passing to moduli, we obtain

$$|x|^3|y-z| + |y|^3|z-x| + |z|^3|x-y| \geq |x-y||y-z||z-x||x+y+z| \tag{12}$$

Let a, b, c, z_P be the affixes of points A, B, C, P, respectively. In (12) consider $x = z_P - a$, $y = z_P - b$, $z = z_P - c$ and obtain inequality (10).

Remarks. 1) If P is the circumcenter O of triangle ABC, after some elementary transformations, (10) becomes

$$\frac{R^2}{6r} \geq OG. \tag{13}$$

2) Squaring both sides of (13), we obtain

$$R^2 \geq 36r^2 \cdot OG^2. \tag{14}$$

Using the relation $OG^2 = R^2 - \frac{1}{9}(\alpha^2 + \beta^2 + \gamma^2)$, (14) is equivalent to

$$R^2(R^2 - 4r^2) \geq 4r^2[8R^2 - (\alpha^2 + \beta^2 + \gamma^2)]. \tag{15}$$

The inequality (15) improves Euler's inequality for the class of obtuse triangles. This is equivalent to proving that $\alpha^2 + \beta^2 + \gamma^2 < 8R^2$ in any such triangle. The last relation can be written as $\sin^2 A + \sin^2 B + \sin^2 C < 2$, or $\cos^2 A + \cos^2 B - \sin^2 C > 0$. That is

$$\frac{1 + \cos 2A}{2} + \frac{1 + \cos 2B}{2} - 1 + \cos^2 C > 0,$$

which reduces to $\cos(A + B)\cos(A - B) + \cos^2 C > 0$. This is equivalent to $\cos C[\cos(A - B) - \cos(A + B)] > 0$, i.e. $\cos A \cos B \cos C < 0$, which is clearly true.

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Proving of Inequalities by Contradiction. Using Inequalities to Solve Equations. Linear Equations. Quadratic Equations.Â Complex number z=a+bi on coordinate pllane xOy is represented by point M with coordinates (a,b). x-axis is called real axis and y-axis is called imaginary axis. Real numbers are represented by points of real axis and purely imaginary numbers are represented by points of imaginary axis. On the figure there are depicted 8 complex numbers: `z_1=2+i`, `z_2=3`, `z_3=2i`, `z_4=-1+i`, `z_5=-2.5`, `z_6=-1-i`, `z_7=-3i`, `z_8=3-2i`. Corresponding points are (2,1), (3,0), (0,2), (-1,1), (-2.5,0), (-1,-1), (0,-3), (3,-2). Not . The ratio is a complex number with argument while the ratio has argument . Hence for the product of these ratios to be real means is a multiple of . In other words, the points lie on a circle.Â [2] T. Andreescu and D. Andrica, Proving some geometric inequalities by using complex numbers, Educatia Matematica Vol. 1, Nr. 2 (2005), pp. 19â€“26. Advertisements.